Lesson: Intro to Three-Dimensional Coordinate Systems

Show how we can draw the three-dimensional coordinate system using the right-hand rule.

How can we consider graphing $f(x, y) = x^2 + y^2$?

Using the (x, y, z) coordinate, what do we have if we set …

- $x = 0$?

- $y = 0$?

- $z = 0$?

How do these 3 coordinate plans divide up the coordinate system?

<u>Example</u> What part of the Cartesian coordinates is each referring to?

a. $x \geq 0$

b. $y = 3$

c. $x \leq 0, y \leq 0, z \leq 0$

Example What points satisfy $x^2 + z^2 = 9$ and $y = 2$?

Lesson: The Standard Form of a Sphere

What is the standard equation for the sphere of radius a and center (x_0, y_0, z_0)?

<u>Example</u> Find the center and radius for the sphere: $4x^2 + 4y^2 + 4z^2 - 8x - 3y = 8$

Lesson: The Distance Between Two Points in 3 Dimensions

State the distance formula.

<u>Example</u> Find the distance between $P_1(4, 5, 6)$ and $P_2(5, 7, 8)$

Examples: Describing the Regions Created by Different Equations

Describe what each set of points look like that fit the following conditions:

a. $x = 4, y = 6$

b. $x^2 + y^2 = 9, z = x$

c. $x^2 + y^2 + z^2 = 9, y = z$

d. $z = y^2, x = 2$

Examples: Describing the Regions Created by Different Equations and Inequalities

Describe what each set of points look like that fit the following conditions:

a. $x \geq 0, y \geq 0, z \geq 0$

b. $x \geq 0, y \geq 0, z = 0$

c. $x^2 + y^2 \leq 4, z = 0$

d. $x^2 + y^2 \leq 4$, no restriction on z

f. $1 \leq x^2 + y^2 + z^2 \leq 9$

Examples: The Standard Form of a Sphere

Find the center C and the radius a for the spheres:

a. $x^2 + y^2 + z^2 + 4x - 6y - 4z = 0$

b. $2x^2 + 2y^2 + 2z^2 - 8y + 6z = 0$

c. $(x-2)^2 + (y-1)^2 + (z+1)^2 = 103 + 4x + 2y - 2z$

Examples: Finding the Equation that Fulfills Given Criteria

a. Find the equation that describes the plane perpendicular to the x-axis at $(5, 0, 0)$

b. Find the equation that describes the plane through the point $(2, -1, 1)$ perpendicular to the y-axis.

c. Find the equation of a circle of radius 3 centered at $(2, -2, 1)$ and lying in a plane parallel to the yz-plane.

Lesson: Intro to Vectors

What is the definition of a vector?

How do we describe and represent a vector with a line segment?

How do we represent vectors?

When are two vectors equal?

How can we represent a vector created by a line segment between two points?

Define the magnitude of a vector:

What vector(s) have a length of 0?

Example Find the component form and length of the vector with initial point $P(1, 2, 3)$ and terminal point $Q(-2, 1, 5)$.

Physics often discuss forces, which intuitively means the push or pull on a object and how that object moves. Forces are just vector quantities. Here is an example: a small cart is being pulled along a smooth horizontal floor with a 10-lb force $\vec{F}$, making a $45°$ angle to the floor. How can we describe the effective force moving the cart forward?

Lesson: Intro to Vector Operations

Suppose we have three vectors: $\vec{u} =< u_1, u_2, u_3 >$, $\vec{v} =< v_1, v_2, v_3 >$, and $\vec{w} =< w_1, w_2, w_3 >$. Let a and b be scalars.

Describe vector addition:

Describe scalar multiplication:

Write out the **Properties of Vector Operations**

Example Let $\boldsymbol{u} =< 1, 2, 3 >$ and $\boldsymbol{v} =< -4, 3, -2 >$. Find the components of the following:

a. $2\boldsymbol{u} + 4\boldsymbol{v}$

b. $\boldsymbol{v} - \boldsymbol{u}$

c. $|\frac{1}{2}\boldsymbol{v}|$

Lesson: Intro to Unit Vectors

What are the standard unit vectors?

How can any vector be written as a linear combination of the unit vectors?

How can we take any vector and rewrite it as a unit vector?

<u>Example</u> Find a unit vector $\boldsymbol{u}$ in the direction of a vector from $P(0,1,2)$ to $Q(3,-1,1)$.

In physics, speed in relation to a vector just refers to the vector's magnitude.

<u>Example</u> If $\boldsymbol{v} = 2\mathrm{i} - \mathbf{3}\mathrm{j}$ is a velocity vector, express $\boldsymbol{v}$ as a product of its speed times its direction of motion.

Summarize the results of this lesson:

Examples: Components and Magnitudes of a Vector

Let $\boldsymbol{u} =< 2, 5 >$ and $\boldsymbol{v} =< 4, -1 >$. Find the component form and magnitude of the following vectors.

a. $4\boldsymbol{v}$

b. $\boldsymbol{u} - \boldsymbol{v}$

c. $\frac{3}{5}\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$

Examples: Finding the Component Form of a Vector

Find the component form of each vector.

a. The vector $\overrightarrow{PQ}$ when $P = (3, 4)$ and $Q = (7, -1)$

b. The vector $\overrightarrow{OP}$ where O is the origin, P is the midpoint of the line segment AB, and $A = (-3, 1)$ and $B = (-1, 5)$

c. The unit vector obtained by rotating the vector $< 1, 0 >$ $135°$ counterclockwise about the origin.

Examples: Finding the Magnitude of Vector Forces

Refer to the drawing in the video. A 50-N weight is suspended by two wires. IF the magnitude of vector $\overrightarrow{F_1}$ is 35N, find the angle α and the magnitude of the vector $\overrightarrow{F_2}$.

Examples: Geometric Representations of Vectors

Copy the vector diagram in the video, and the perform the following operations geometrically.

a. $\boldsymbol{u} + \boldsymbol{v}$

b. $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w}$

c. $2\boldsymbol{u} - \boldsymbol{v}$

Examples: Expressing Vectors as a Product of Length and Direction

Express each vector as a product of its length and direction.

a. $3\boldsymbol{i} - 2\mathbf{j} + 5\mathbf{k}$

b. $\frac{1}{2}\boldsymbol{i} + \frac{1}{4}\boldsymbol{k}$

Lesson: The Dot Product

How can we find the angle between two vectors?

Part 1: when we can utilize a right triangle

Part 2: when we cannot utilize a right triangle

Define the Dot Product:

Summarize the angle between vectors:

What is another way to define the dot product?

Example Find the dot product.
a. $\langle 3, -2, 5\rangle \cdot \langle -1, 3, 1\rangle$

b. $(\frac{1}{2}\boldsymbol{i} - \frac{1}{3}\boldsymbol{j} + \frac{1}{4}\boldsymbol{k}) \cdot (6\boldsymbol{i} + 6\boldsymbol{j} - 4\boldsymbol{k})$

Example Find the angle between the vectors $\boldsymbol{u} = \langle 2, 1, 3\rangle$ and $\boldsymbol{v} = \langle -1, 0, 3\rangle$

Define what it means for two vectors to be orthogonal.

<u>Example</u> Determine if the following vectors are orthogonal: $\boldsymbol{u} = \langle 2, 3\rangle$ and $\boldsymbol{v} = \langle -6, 4\rangle$

What vector is orthogonal to every nonzero vector?

List the Properties of the Dot Product

Lesson: Vector Projections

Explain how vector projections work:

Summarize the vector projection of $\vec{u}$ onto $\vec{v}$:

Summarize the scalar component of $\vec{u}$ in the direction of $\vec{v}$L

Example Let $\boldsymbol{u} = 3\boldsymbol{i} - 2\mathbf{j} + \mathbf{k}$ and $\boldsymbol{v} = 2\boldsymbol{i} + \mathbf{j} - 3\mathbf{k}$.

a. Find $\boldsymbol{u} \cdot \boldsymbol{v}$, $|\boldsymbol{u}|$, $|\boldsymbol{v}|$

b. Find the cosine of the angle between the vectors.

c. Find the scalar component of $\boldsymbol{u}$ in the direction $\boldsymbol{v}$.

d. Find the vector $proj_{\vec{v}}\vec{u}$

Lesson: Work and Vectors

With vectors, we can study work done by a constant force when a force is applied in the direction other than the direction in motion.

What is the equation for work done by a constant force applied in the direction of motion? Explain this equation.

Explain how we can understand a force moving on an object through a displacement vector.

Example If $|\boldsymbol{F}| = 20N$ and $|\boldsymbol{D}| = 2m$ and $\theta = 60°$, find the work done by $\boldsymbol{F}$.

Examples: Dot Products and Projections

For the following vectors, find
i) $\boldsymbol{u} \cdot \boldsymbol{v}$, $|\boldsymbol{u}|$, $|\boldsymbol{v}|$
ii) the angle between the vectors
iii) the scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$
iv) The vector $proj_v \boldsymbol{u}$

a. $\boldsymbol{u} = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

b. $\boldsymbol{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}} \rangle, \boldsymbol{v} = \langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}} \rangle$

Examples: Working with Diagonals of Shapes

Show that the diagonals of a rhombus are perpendicular.

Examples: Work

How much work does it take to slide a crate 10m along a loading dock by pulling on with a 100N force at an angle of $30°$ from the horizontal?

Lesson: The Cross Product

Define the cross product and where it comes from:

When are two vectors parallel?

List the properties of the cross product:

Explain the significance of $|\vec{u} \times \vec{v}|$

How can you use the determinant formula for calculate the cross product?

Example Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ If $\vec{u} = \langle 2, 1, 3 \rangle$ and $\vec{v} = \langle -1, 3, 1 \rangle$

Example Find a vector perpendicular to the plane of $P(1,0,1), Q(2,1,-1), R(-1,1,2)$

Example Find the area of the triangle with vertices $P(1,0,1), Q(2,1,-1), R(-1,1,2)$

Example Find a unit vector perpendicular to the plane of $P(1,0,1), Q(2,1,-1), R(-1,1,2)$

Explain how torque relates to vectors:

<u>Example</u> Find the magnitude of the force **F** at the pivot point P in the picture.

Explain the triple scalar or box product:

Summarize how to calculate the triple scalar product as a determinant:

<u>Example</u> Find the volume of the box determined by $\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{v} = 5\boldsymbol{i} + 2\boldsymbol{k}, \boldsymbol{w} = -2\boldsymbol{j} + \boldsymbol{k}$

Review: Determinants

Find the following determinants.

a. $\begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix}$

b. $\begin{vmatrix} -1 & 6 \\ 3 & 5 \end{vmatrix}$

c. $\begin{vmatrix} 3 & 1 & -1 \\ 2 & 1 & 2 \\ 1 & 4 & 2 \end{vmatrix}$

Compute $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{vmatrix}$ two ways.

Lesson: Triple Scalar or Box Product

Explain the Triple Scalar or Box Product

Summarize how to calculate the triple scalar product as a determinant:

Example Find the volume of the box determined by $\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{v} = 5\boldsymbol{i} + 2\boldsymbol{k}, \boldsymbol{w} = -2\boldsymbol{j} + \boldsymbol{k}$

Examples: Finding Area of Triangles with the Cross Product

Find i) the area of the triangle determined by the points P, Q, and R. ii) Find a unit vector perpendicular to plane PQR.

$P(1, 2, 1), Q(1, 0, -1), R(3, -1, 1)$

Lesson: Lines and Line Segments in Space

Explain the vector equation for a line.

What are the parametric equations for a line?

Example Find the parametric equations for the line through $(-1, 0, 2)$ parallel to $\boldsymbol{v} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

Example Find the parametric equations for the line through $P(-2, 0, 1)$ and $Q(-1, 1, 2)$

Example Parametrize the line segment joining the points $P(-2, 0, 1)$ and $Q(-1, 1, 2)$.

How do you find the distance from a point S to a line through P parallel to a vector $\boldsymbol{v}$**?**

Lesson: Planes in Space

How can we find the equation for a plane in space?

Example Find an equation for the plane through $P_0(-1, 2, 3)$ perpendicular to $\boldsymbol{n} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

Example Find an equation for the plane through $A(0,0,2), B(3,0,0), C(0,1,0)$

Example Find a vector parallel to the line of intersection of the planes $2x-3y-6z=12$ and $2x+2y-z=6$.

Example Find parametric equations for the line in which the planes $2x - 3y - 6z = 12$ and $2x + 2y - z = 6$ intersect.

Example Find the point where the line:

$$x = \frac{3}{2} + 2t, y = 2t, z = 1 - t$$

Intersects the plane $2x - 3y - 6z = 12$

Describe the distance from a point S to a plane with normal **n** at point P.

<u>Example</u> Find the distance from $S(1, 2, 1)$ to the plane $2x - 3y + 6z = 12$.

Example Find the angle between the planes $2x - 3y - 6z = 12$ and $2x + 2y - z = 6$.

Examples: Parametric Equations for Lines

Find parametric equations for the lines.

a. The line through $P(1, 1, 2)$ and $Q(-1, 0, 1)$

b. The line through $(2, 4, 6)$ perpendicular to the plane $x - y + 4z = 8$.

Examples: Equations for Planes

Find the equations for the planes.

a. The plane through $(1, 1, 2), (3, 0, 3), (0, -1, 1)$

b. The plane through $A(1, -1, 0)$ perpendicular to the vector from the origin to A.

Examples: Finding Planes Containing Intersecting Lines

Find the plane containing the intersecting lines:

$$L_1: x = 1 - t, \qquad y = 3 + t, \qquad z = 1 + t, \qquad -\infty < t < \infty$$
$$L_2: x = 1 - 2s, \qquad y = 1 + 4s, \qquad z = 3 - 3s, \qquad -\infty < t < \infty$$

Examples: Distance from a Point to a Line

Find the distance from the point $(0, 1, 1)$ to the line $x = 2 + 3t, \;\; y = 1 - t, \;\; z = -2 + 2t$

Examples: Sketching Quadric Surfraces

Sketch the following.

a. $x^2 + z^2 = 9$

b. $z = 4 + x^2 + y^2$

c. $z = 1 - y^2 + x^2$

Lesson: Intro to Vector-Valued Functions

What are component functions?

What is a vector-valued function?

Draw an example of a vector-valued function:

Examples: Determining a Space Curve from Its Function

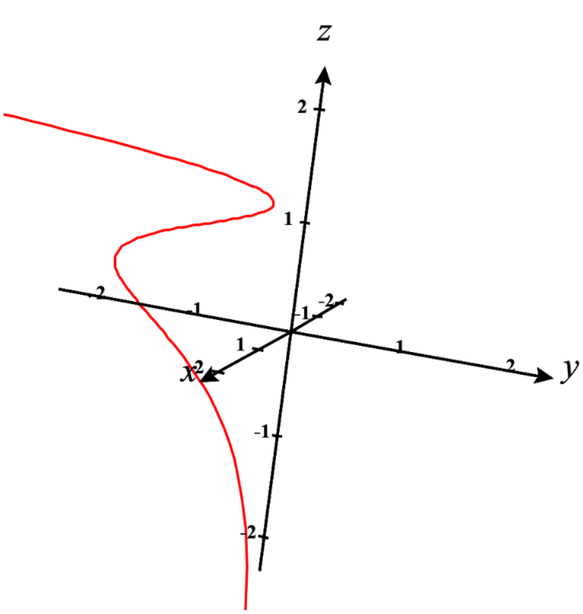
Determine which $\vec{r}(t)$ matches the space curves below.

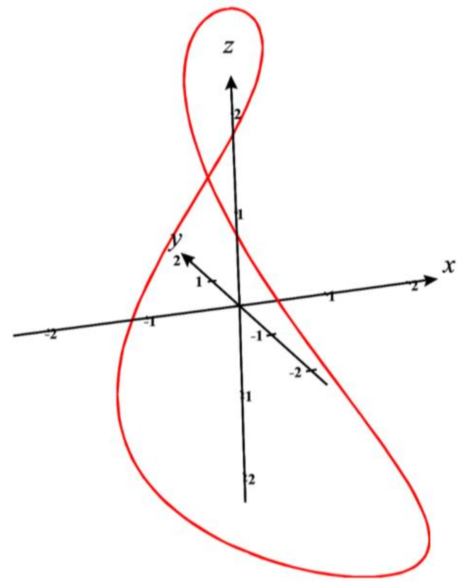
a. $\vec{r}(t) = t\vec{\imath} - (\sin t)\vec{\jmath} + (\ln t)\vec{k}$

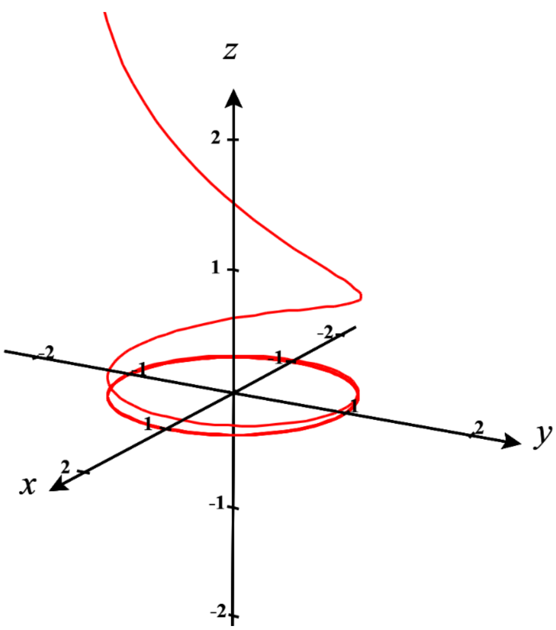
b. $\vec{r}(t) = (\sin 2t)\vec{\imath} + (\cos 2t)\vec{\jmath} + (e^{t})\vec{k}$

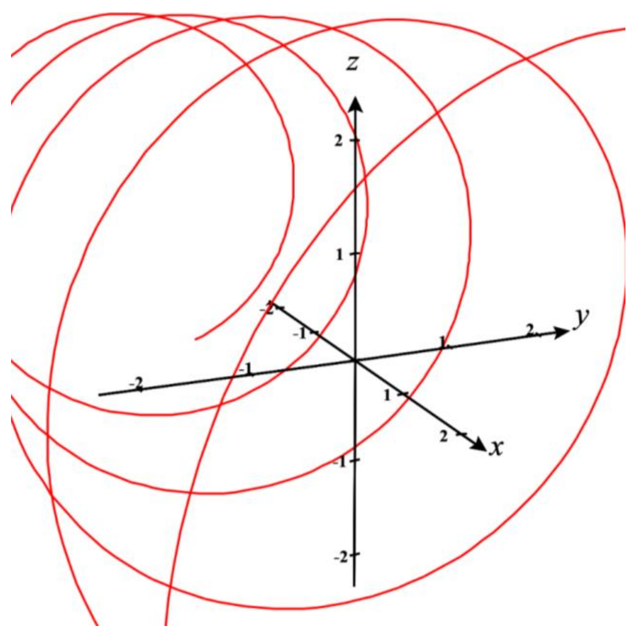
c. $\vec{r}(t) = \left(\frac{1}{2}t\right)\vec{\imath} + (3\cos t)\vec{\jmath} + (3\sin t)\vec{k}$

d. $\vec{r}(t) = (\sin 2t)\vec{\imath} + (3\cos t)\vec{\jmath} + (3\sin t)\vec{k}$









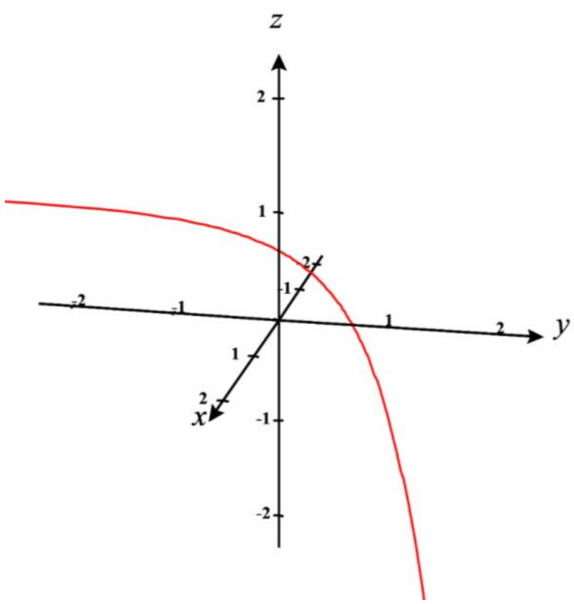
Determine which $\vec{r}(t)$ matches the space curves below.

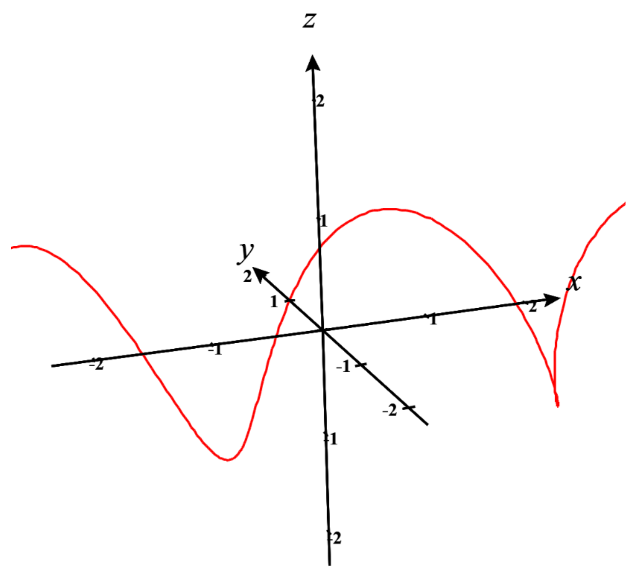
a. $\vec{r}(t) = (\sin t)\,\vec{\imath} + (\cos t)\vec{\jmath} + (\frac{1}{2}t)\vec{k}$

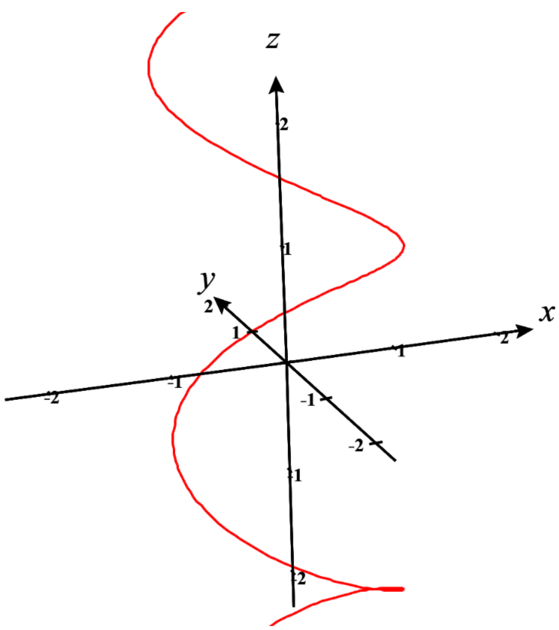
b. $\vec{r}(t) = \left(\frac{1}{2}t\right)\vec{\imath} + (\cos t)\vec{\jmath} + (\sin t)\vec{k}$

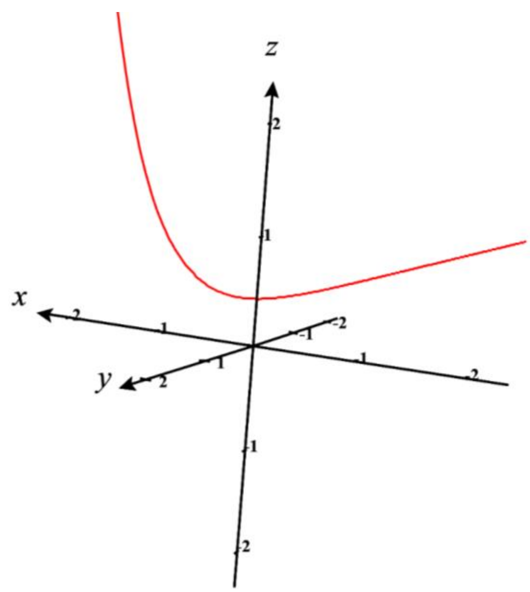
c. $\vec{r}(t) = (t)\vec{\imath} + (\ln t)\vec{\jmath} + (1)\vec{k}$

d. $\vec{r}(t) = (1)\vec{\imath} + (t)\vec{\jmath} + (e^{t})\vec{k}$









Examples: Limits with Vectors

Find the following limits.

a. $\lim_{t\to\frac{\pi}{2}}[(\cos t)\vec{\imath} + (\sin 2t)\vec{\jmath} + \tan(4t)\vec{k}]$

b. $\lim_{\mathrm{t}\to 1}\left[\frac{t-1}{\sqrt{t}-1}\vec{\imath} + \frac{t^3-1}{\ln t}\vec{\jmath} - (\tan^{-1} t)\vec{k}\right]$

Lesson: The Basics of Tangent Lines for Space Curves

Write out what is reviewed in the beginning of the video.

Show how to find the tangent line for $\vec{r}(t) = (\sin t)\vec{\imath} + (\cos t)\vec{\jmath} + (t)\vec{k}$ at $t = 0$

Examples: Differentiating Vector-Valued Functions

Below $\vec{r}(t)$ represents the position of a particle in space at time time. Find i) the particle's velocity, ii) the particle's acceleration, iii) the speed of the particle at any moment, and iv) find a way to write the particle's velocity as a product of its speed and direction

a. $\vec{r}(t) = (t^2)\vec{\imath} + (\cos t)\vec{\jmath} - (e^{-t})\vec{k};\ t = 0$

b. $\vec{r}(t) = (\sec t)\vec{\imath} - (\ln 4t)\vec{\jmath} + (\tan t)\vec{k};\ t = \frac{\pi}{4}$

Derivative Rules Challenge for Calculus III

Name	Function Form	Derivative
Product Rule	$f(x) * g(x)$	
Quotient Rule	$\frac{f(x)}{g(x)} \ (g(x) \neq 0)$	
Sine	$\sin x$	
Cosine	$\cos x$	
Tangent	$\tan x$	
Cotangent	$\cot x$	
Secant	$\sec x$	
Cosecant	$\csc x$	
Chain Rule	$f\left(g(x)\right)$	
Natural Log	$\ln x$	
Natural Exponent	e^x	
Hyperbolic sine	$\sinh x$	
Hyperbolic cosine	$\cosh x$	

Find the derivatives:

a. $y = \cos 5t$

b. $y = e^{-t}$

c. $y = t \sin t$

d. $y = e^t \cos t$

e. $y = \frac{t^2-1}{\ln t}$

f. $y = \cos^3 t$

g. Find the second derivative of $y = \cos^3 t$

Examples: Finding Tangent Lines of Vector-Valued Functions

Find the tangent lines to the smooth curves at the given moment t_0. Write the line with parametric equations.

a. $\vec{r}(t) = (t^2)\vec{\imath} + (4t+1)\vec{\jmath} + (t^4)\vec{k},\ t_0 = 2$

b. $\vec{r}(t) = (t \ln t)\vec{\imath} - (\ln 2t)\vec{\jmath} + \left(\frac{t-1}{t+3}\right)\vec{k},\ t_0 = 1$

Examples: Finding Tangent Lines Given a Point

Find the value(s) of t so that the tangent line to the given curve contains the given point.

$\vec{r}(t) = (t)\vec{\imath} - \left(\frac{2}{3}t^{\frac{3}{2}}\right)\vec{\jmath} + (5)\vec{k}$, $(0, -\frac{8}{3}, 5)$

Examples: Integrating Vector-Valued Functions

Integrate the following.

a. $\int_1^2 \left[(4-4t)\vec{\imath} + \left(6\sqrt{t}\right)\vec{\jmath} + \left(\frac{4}{t}\right)\vec{k} \right] dt$

b. $\int_0^{\frac{\pi}{2}} \left[(\cos t)\vec{\imath} + (t \sin t)\vec{\jmath} + (\sin^2 t)\vec{k} \right] dt$

Review: Some Common Integration Techniques and Tricks for Calc III

Trig Identities

Pythagorean Identities

Half Angle Identities + Modifications

a. $\int \tan^2 x \ dx$

b. $\int \sin^2 x \ dx$

Substitution

a. $\int 2x\sqrt{x^2+1}dx$

b. $\int t\ e^{t^2} dt$

Integration by Parts

Formula:

a. $\int x \sin x \; dx$

b. $\int \ln x \; dx$

Examples: Solving Initial Value Problems with Vector-Valued Functions

Solve the initial value problem.

$$\frac{d\vec{r}}{dt} = t\vec{\imath} + \frac{1}{2}t\vec{\jmath} + t^3\vec{k}$$

$$\vec{r}(0) = \vec{\imath} - \vec{\jmath}$$

Lesson: Arc Length Parameter

We are going to take interest in different mathematical features of curves that will describe how they turn or twist.

First we will discuss this idea conceptually. Do your best to summarize the difference between arc length versus arc length parameter.

Write the arc length formula for a smooth curve $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$, $a \le t \le b$ that is traced only once as t increases from a to b:

What else does this formula represent?

How could we rewrite this formula in short hand?

Example Find the length of the path from $t = 0$ to $t = \pi$ for $\vec{r}(t) = (\sin 2t)\vec{\imath} + (\cos 2t)\,\vec{\jmath} + (t)\vec{k}$

Now write out how to compute arc length parameter:

Example Find the arc length parameter for the previous example from $t = 0$ to $t = \pi$ for $\vec{r}(t) = (\sin 2t)\vec{\imath} + (\cos 2t)\vec{\jmath} + (t)\vec{k}$

What are the pros and cons of arc length versus arc length parameter?

Lesson: The Unit Tangent Vector

How do we define the unit tangent vector?

Explain the derivation for an easier way to compute this formula.

Examples: Arc Length Parameter

Find the arc length parameter along the curve from the point where t=0 by evaluating:

$$s = \int_0^t \{\vec{v}(\tau) d\tau$$

Then find the length of the indicated portion of the curve.

$$\vec{r}(t) = (1 + 2t)\vec{\imath} + (1 + 4t)\vec{\jmath} + (1 + 8t)\vec{k}, -1 \le t \le 0$$

Examples: Finding Tangent Vectors and Lengths

Find the curve's unit tangent vector, then find the length of the indicated portion of the curve.

a. $\vec{r}(t) = (4\cos t)\vec{\imath} - (4\sin t)\vec{\jmath} + (3t)\vec{k}$, $0 \le t \le \pi$

b. $\vec{r}(t) = (t\sin t)\vec{\imath} - (t\cos t)\vec{\jmath}$, $0 \le t \le \pi$

Lesson: Curvature and Normal Vectors

How do we define curvature?

Explain the idea of curvature conceptually:

What is one alternative way to derive an easier formula for curvature?

Example Find the curvature κ for the curve $\vec{r}(t) = (\cos t + t \sin t)\vec{\imath} + (\sin t - t \cos t)\vec{\jmath}, t > 0$

Example Find the curvature of a circle $\vec{r}(t) = (a \cos t)\vec{\imath} + (a \sin t)\vec{\jmath}$ for a circle with radius a

Define a principal unit normal vector.

Explain the principal normal vector conceptually.

Show how to derive a simpler formula for calculating $\vec{N}$

Example Find the normal vector $\vec{N}$ for the curve $\vec{r}(t) = (\cos t + t\sin t)\vec{\imath} + (\sin t - t\cos t)\vec{\jmath}, t > 0$

Examples: Finding $\vec{T}$, κ, and $\vec{N}$ for Space Curves

Find $\vec{T}$, κ, and $\vec{N}$ for the space curves:

a. $\vec{r}(t) = (e^t \cos t)\vec{\imath} + (e^t \sin t)\vec{\jmath} + 5\vec{k}$

b. $\vec{r}(t) = (\cos^3 t)\vec{\imath} + (\sin^3 t)\vec{\jmath}$, $0 < t < \frac{\pi}{2}$

Examples: Maximum Curvature

Find the maximum curvature for the function $f(x) = \ln x$

Examples: Writing Acceleration with Tangential and Normal Components

Write $\vec{a} = a_T\vec{T} + a_N\vec{N}$ without finding $\vec{T}$ and $\vec{N}$ for

$$\vec{r}(t) = (t\cos t)\vec{\imath} + (t\sin t)\vec{\jmath} + t^2\vec{k}$$

Examples: Finding $\vec{B}$ and τ

In section 13.4, you found $\vec{T}$, κ, and $\vec{N}$ for the space curves. Now find $\vec{B}$ and τ.

a. $\vec{r}(t) = (e^t \cos t)\vec{\imath} + (e^t \sin t)\vec{\jmath} + 5\vec{k}$

b. $\vec{r}(t) = (\cos^3 t)\vec{\imath} + (\sin^3 t)\vec{\jmath}$, $0 < t < \frac{\pi}{2}$

Examples: Find and Sketch the Domain for Multivariable Functions

Find the sketch the domain.

a. $f(x) = \ln\left(x^2 + y^2 - 4\right)$

b. $f(x) = \sqrt{(x^2 - 1)(x^2 - 4)}$

Examples: Sketching Level Curves and Level Surfaces

Find the level curves or level surfaces for the following. Do your best to sketch or at least describe the region for surfaces.

a. $f(x, y) = \sqrt{x^2 + y^2}$

b. $f(x, y) = 1 - |x| - |y|$

c. $f(x, y, z) = \ln{(x^2 + y^2 + z^2)}$

d. $f(x, y, z) = z - x^2 - y^2$

Review: Algebra Tricks for Limits in 1 Variable vs Multivariables

1. Factoring to Cancel

$\lim\limits_{x \to 1} \frac{x-1}{x^2-1}$

Crashing Course Factoring Review:

Factor by the GCF: $\frac{3x^2-5x^3}{x^5-2x^4+3x^3+2x^2}$

Factor by Grouping:

a. $ax + ay + bx + by$

b. $\frac{x-1}{xy-y-2x+2}$

Factor Trinomials:

a. $x^2 + 5x + 6$

b. $2x^2 + 5x + 3$

c. $15x^2 + 14x - 8$

Difference of Cubes Formulas

Difference of Cubes with Polynomial Long Division: $(x^3 + 1) \div (x + 1)$

2. Using the Conjugate

What is the conjugate?

Show how to multiply by the conjugate in this one-variable example:

$\lim_{x \to 1} \frac{x-1}{\sqrt{x+3}-2}$

3. Two Common Trig Tricks

Write the two common limits for trig:

$\lim_{x \to 0} \frac{\tan 3x}{x}$

Examples: Finding Limits of Multivariable Functions Using Algebra and Trig Tricks

a. $\lim\limits_{(x,y)\to(0,0)} \frac{e^x \sin y}{y}$

b. $\lim\limits_{(x,y)\to(2,-5)} \frac{y+5}{x^2y-xy+5x^2-5x}$

c. $\lim\limits_{(x,y)\to(3,0)} \frac{\sqrt{3x-y}-3}{3x-y-6}$

Examples: Limits that Do Not Exist

Show that the limits do not exist.

a. $\lim_{(x,y)\to(0,0)} \frac{x+\sin y}{y+\sin x}$

b. $\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$

Examples: Second-Order Partial Derivatives

Find all the second order partial derivatives.

a. $f(x, y) = xy^2 + \cos xy + y \sin x$

b. $f(x, y) = \ln(x + y)$

c. $z = xe^{\frac{y}{x^2}}$

Examples: The Chain Rule for Multivariable Functions

Find the derivatives, and then determine the value at the given point.

a. $\frac{dw}{dt}$ for $w = \ln(x^2 + y^2 + z^2)$ where $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$ at the point $t = 3$

b. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ for $u = e^{qp} \sin^{-1} r$ where $p = \frac{1}{z}, q = y^2 \ln z, r = \sin x$ at point $(x, y, z) = \left(\frac{\pi}{4}, 1, 1\right)$

Examples: Implicit Differentiation, Again

Find $\frac{dy}{dx}$ at the given point: $ye^x + \sin xy + x - \ln 5 = 0, (\ln 5, 0)$

Examples: Finding Direction Derivatives

Find the derivative of the function P in the direction of $\vec{u}$

a. $f(x,y) = \frac{x-y}{xy+2}, P(1,-1), \vec{u} = 12\vec{i} + 5\vec{j}$

b. $f(x,y,z) = \cos xy + e^{yz} + \ln zx\,, P\left(1,0,\frac{1}{2}\right), \vec{u} = \vec{i} + 2\vec{j} + 2\vec{k}$

Examples: Directions Where a Function Increases or Decreases Most Rapidly

Where does $f(x, y, z) = xe^y + z^2$ increase or decrease most rapidly at $P(4, 1, 1)$? Find the derivative at these directions.

Examples: Tangent Lines to Level Curves

Sketch the curve with the gradient and the tangent line at the given point. Write the equation for the tangent line.

$$x^2 + y^2 = 9, (\sqrt{3}, \sqrt{3})$$

Examples: Tangent Planes and Normal Lines to Surfaces

Find equations for the tangent plane and normal line at the point P on the surface:

$$x^2 + y^2 - 2xy - x + 3y - z = -4, P(2, -3, 18)$$

Examples: Tangent Lines to Intersecting Surfaces

Find parametric equations for the line tangent to the curve of intersection of the surface at the given point:

Surfaces: $x^2 + 2y + 2z = 4; y = 1; P\left(1,1,\frac{1}{2}\right)$

Lesson: Extreme Values and Saddle Points

Let $f(x, y)$ be defined on a range R containing the point (a, b). Then:

- $f(a, b)$ is a **local maximum** if:

- $f(a, b)$ is a **local minimum** if:

State the First Derivative Test for Local Extreme Values:

Define a **critical point** for $f(x, y)$

What is a saddle point?

What is the **discriminant** or **Hessian** of $f(x, y)$?

The Second Derivative Test for Local Extreme Values

Let $f(x, y)$ be a function where the first and second derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then:

i. f has a local maximum at (a, b) if:

ii. i. f has a local minimum at (a, b) if:

iii. f has a saddle point at (a, b) if:

iv. the test is inconclusive if:

<u>Examples</u> Find the local extreme values of the following functions.

a. $f(x, y) = -5x^2 - 2y^2 + 2xy + 4x + 4y - 4$

b. $f(x, y) = e^x - xe^y$

c. $f(x, y) = 1 - \sqrt[3]{x^2 + y^2}$

Examples: Finding Local Extrema

Find all the local maxima, local minima, and saddle points

a. $f(x,y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$

b. $f(x, y) = \ln(x + y) + x^2 - y$

Examples: Finding The Critical Points for Interesting Systems

Find *only* the critical points for the following.

a. $f(x,y) = x^3 - y^3 + 3xy - 6$

b. $f(x,y) = 8xy - x^4 - y^4$

c. $f(x, y) = e^{x^2+y^2-8x}$

d. $f(x, y) = \frac{1}{x^2+y^2-4}$

Examples: Finding Absolute Extrema

Find the absolute maxima and minima for:

$f(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines:

$x = 0, y = 4, y = x$

Examples: Lagrange Multipliers with Two Independent Variables and One Constraint

a. Find the extreme values of $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2 + y^2 - 10 = 0$

b. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

Examples: Lagrange Multipliers with Three Independent Variables and One Constraint

Find the point on the sphere $x^2 + y^2 + z^2 = 9$ farthest from the point $(1, -1,\ 1)$

b. $f(x, y) = \ln(x + y) + x^2 - y$

Examples: Evaluating Iterated Integrals

Evaluate the following integrals.

a. $\int_0^2 \int_0^4 \left(\frac{x}{4} + \sqrt{y}\right) dx\, dy$

b. $\int_1^3 \int_1^{e^2} \frac{\ln x}{xy}\, dx\, dy$

c. $\int_1^2 \int_1^2 x \ln y \; dy \, dx$

Examples: Evaluating Double Integrals Over Rectangles

Evaluate the following integrals over the region R.

a. $\int\int_R y\sin(x+y)\,dA,\ R: 0 \le x \le \pi, -\pi \le y \le 0$

b. $\iint_R \frac{xy^3}{x^2+1}\,dA, R: 0 \le x \le 1, 0 \le y \le 1$

Examples: Sketching Regions of Integration

Sketch the describe regions of integration.

a. $0 \le x \le 4, 0 \le y \le 3x$

b. $0 \le x \le 1, e^x \le y \le e$

Examples: Write an Integral Over a Region

Write an iterated integral for the region R using vertical cross sections AND horizontal cross sections.

a. $y = x^2, y = 2x$

b. $y = 3x, x = 2$

Examples: Sketch the Region of Integration and Evaluate the Integral

Sketch the region of integration and evaluate the integral.

a. $\int_0^{\pi}\int_0^{\cos x} y\,dy\,dx$

b. $\int_0^1\int_0^{x^2} 3x^3 e^{xy}\,dy\,dx$

Examples: Sketch the Region of Integration and Reverse the Order of Integration

Sketch the region of integration and write an equivalent double integral with the order of integration reversed.

a. $\int_0^1 \int_{1-x}^{1-x^2} dy\, dx$

b. $\int_0^3 \int_1^{e^y} (x+y) dx\, dy$

Examples: Volume Beneath a Surface

Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$

Examples: Area by Double Integrals

Sketch the region bounded by the given lines and curves, Then express the region's area as a double integral, and integrate.

a. The parabola $x = y - y^2$ and the line $y = x$

b. The lines $y = x, y = \frac{x}{2}$,and $y = 2$

Examples: Integrating with Polar Coordinates

Change the Cartesian integral into an equivalent polar integral, then integrate.

a. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx\, dy$

b. $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{2}{1+\sqrt{x^2+y^2}}\, dy\, dx$

c. $\int_0^1 \int_x^{\sqrt{1-x^2}} (x + 2y) dy\, dx$

Examples: Evaluating Triple Integrated Integrals

Integrate the following.

a. $\int_0^1 \int_0^{1-x^2} \int_3^{9-x^2-y^2} x\, dz\, dy\, dx$

b. $\int_0^1 \int_1^{\sqrt{e}} \int_1^{e} x\, e^x \ln y \frac{(\ln z)^2}{z}\, dz\, dy\, dx$

Examples: Reversing the Order for Triple Integrals

Using the integral: $\int_{-1}^{1}\int_{x}^{2}\int_{0}^{1-x} dz\, dy\, dx$

Rewrite an equivalent integral in the following order:

a. $dy\, dz\, dx$

b. $dy\, dx\, dz$

c. $dz\, dx\, dy$

Examples: Volume with Triple Integrals

Find the volume in the following situations.

a. The region between the cylinder $z = x^2$ and the xy-plane that is bounded by the planes $x = -1, x = 1, y = 0, y = 1$.

b. The region between the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder

$x = 4 - y^\wedge 2$

Examples: Sketching Graphs in Cylindrical Coordinates

Sketch the graph describe by the cylindrical coordinates.

a. $z = r$

b. $r = \theta$

c. $0 \leq r \leq 2\sin\theta, 1 \leq z \leq 2$

Examples: Sketching Graphs in Spherical Coordinates

Sketch the graph describe by the spherical coordinates.

a. $\rho = 4$

b. $\rho \cos \theta = 2$

c. $0 \leq \rho \leq 1, \frac{\pi}{2} \leq \phi \leq \pi, 0 \leq \theta \leq \pi$

Examples: Setting Up Triple Integrals Across Coordinate Systems

Set up the triple integrals for the volume of the sphere $\rho = 4$ in:

a. Spherical coordinates

b. Cylindrical coordinates

c. Rectangular coordinates

Now let D be a solid ball of radius 4 units that is cut by a plane that is 2 units from the center of the sphere. Express the volume of this situation in:

a. Cylindrical coordinates

b. Spherical coordinates

c. Rectangular coordinates

Examples: Double Integral Substitutions

Evaluate $\int_0^2 \int_{\frac{y}{2}}^{\frac{y+4}{2}} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$ using the transformation $x = u + \frac{v}{2}, y = v$.

Evaluate $\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dy\, dx$ using the transformation $x = u^2 - v^2, y = 2uv$.

Not included (no notes file):

823 (Field Trip on CalcPlot3D for Space Curves)