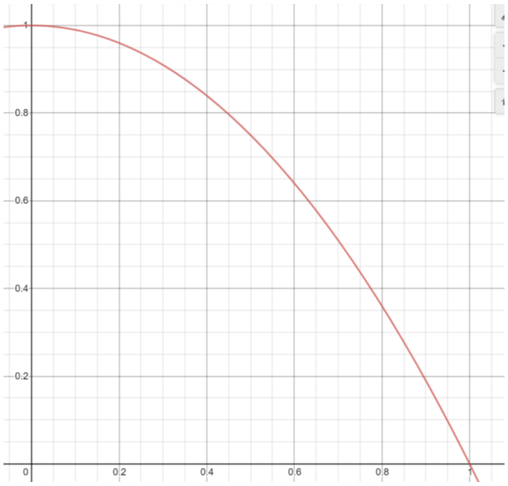
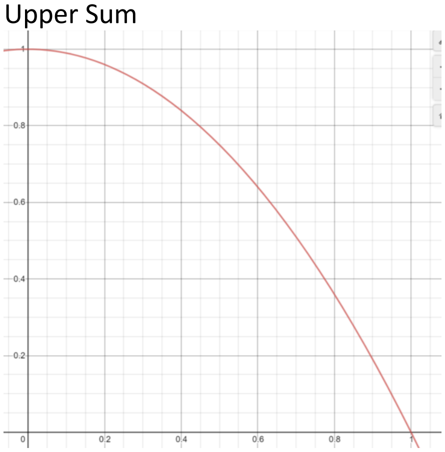
Area and Estimating with Finite Sums

First we will try to understand the idea behind this. Draw out one way to find the area of the curve $y = 1 - x^2$ below.

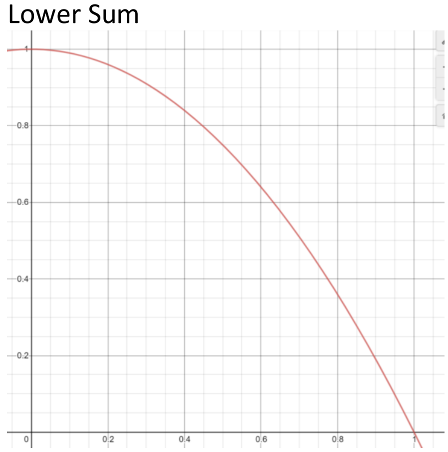


Now we need more formality around this idea. Show the idea of how to create sums using a more algorithmic approach. Detail what each sum means:

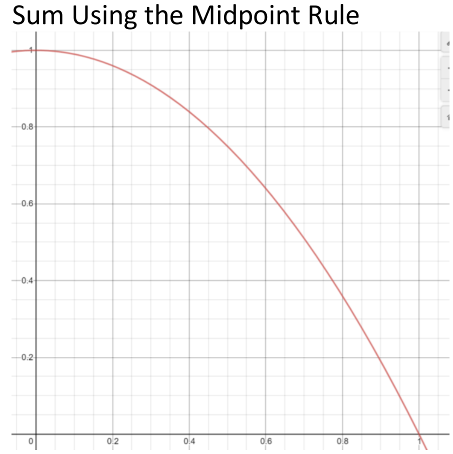
Upper Sum



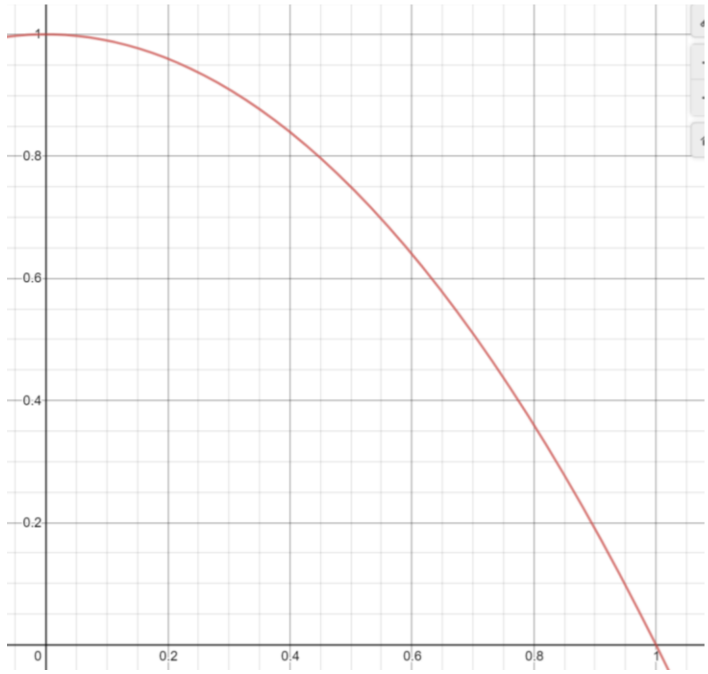
Lower Sum



Sum Using the Midpoint Rule



Now we need some vocabulary. Separate the graph in 4 parts and then label it with the correct terminology:



Define a Riemann Sum:

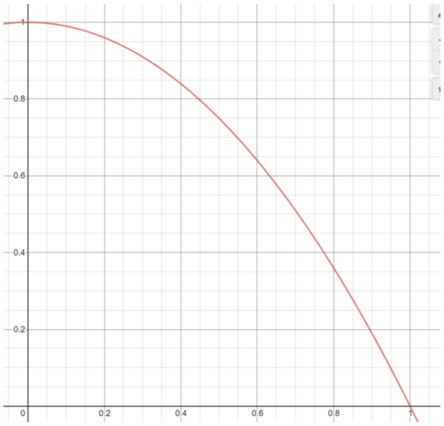
What is a left Riemann sum?

What is a right Riemann sum?

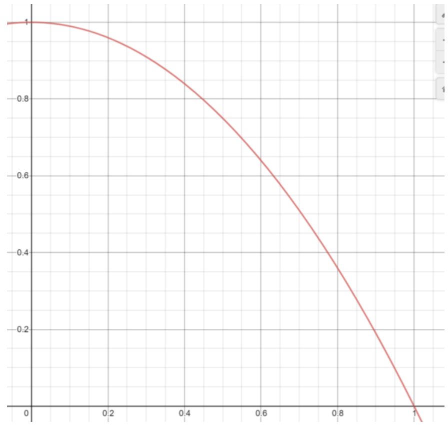
What is a midpoint Riemann sum?

Show how you can make a more precise sum:

8 Subintervals

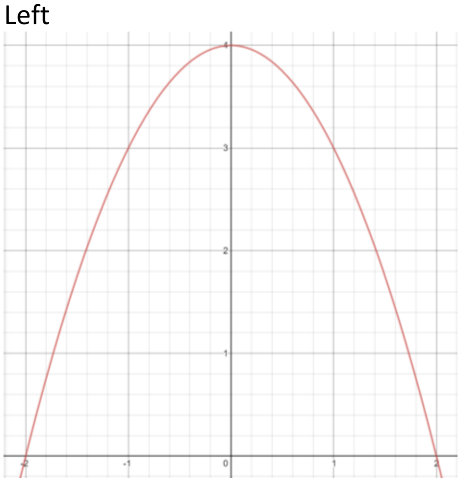


16 Subintervals

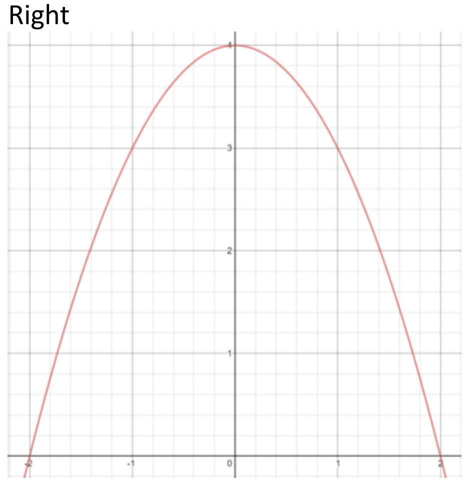


<u>Example</u> Compute the left, right, and midpoint Riemann sums for $f(x) = 4 - x^2$ between $[-2, 2]$ using 4 subintervals

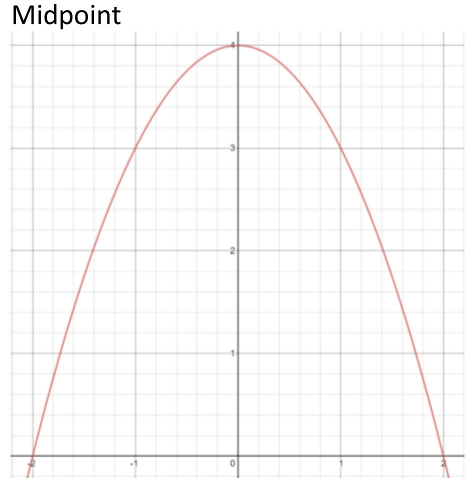
Left



Right



Midpoint



Graphs were created using Desmos.com. They have the greatest graphing software for free, check them out!

Examples of Riemann Sums

Compute the left, right, and midpoint Riemann sum for $f(x) = 3 + x$ from $[1,3]$ using 4 subintervals.

Examples of Riemann Sums – When It Goes Negative?

Compute the left, right, and midpoint Riemann sum for $f(x) = x^3$ from $[-2,2]$ using 4 subintervals.

Examples: Different Ways to Compute Area Estimations

Using $f(x) = -x^2 + 2x + 3$, compute an area estimate using 4 subintervals from x=-1 to x=3.

Compute this estimate using an upper, lower, left-hand, and right-hand sum.

Lesson: The Basics of Sigma Notation

Explain what sigma notation looks like and what it means:

<u>Examples</u> Compute the following sums:

a. $\Sigma_{k=1}^{4} k$

b. $\Sigma_{i=3}^{7} (2i - 1)$

c. $\Sigma_{k=1}^{4} (-1)^{k+1} k$

We have a few handy rules for sigma notation:

1. Sum Rule:

2. Difference Rule:

3. Constant Multiple Rule:

4. Constant Value Rule:

<u>Examples</u> Use the rules above to help calculate the following sums:

a. $\Sigma_{k=1}^{3}(k+5)$

b. $\Sigma_{k=1}^{4}(-3k-2)$

Lesson: Using Sigma Notation and Sum Formulas to Calculate Larger Sums

We've already discussed these basic algebraic rules for sums:

Sum Rule

$$\Sigma_{k=1}^{n}(a_k + b_k) = \Sigma_{k=1}^{n}a_k + \Sigma_{k=1}^{n}b_k$$

Difference Rule

$$\Sigma_{k=1}^{n}(a_k - b_k) = \Sigma_{k=1}^{n}a_k - \Sigma_{k=1}^{n}b_k$$

Constant Multiple Rule

$$\Sigma_{k=1}^{n} ca_k = c * \Sigma_{k=1}^{n}a_k$$

Constant Value Rule

$$\Sigma_{k=1}^{n} c = n * c$$

Now we will add on a few other handy formulas for sums:

The First n Terms

The First n Squares

The First n Cubes

These formulas are particularly handy for larger sums.

Examples Sum the following using the sum formulas:

a. $\Sigma_{k=1}^{15} 3k^2$

b. $\Sigma_{i=1}^{52} (i^3 + 4i - 2)$

c. $\Sigma_{k=1}^{35} k(2k + 5)$

Lesson: Sigma Notation with Riemann Sums

Our goal in this lesson is to figure out how to use Riemann Sums with Sigma Notation. To get you warmed up for this idea, we will practice figuring out how to write certain Riemann sums:

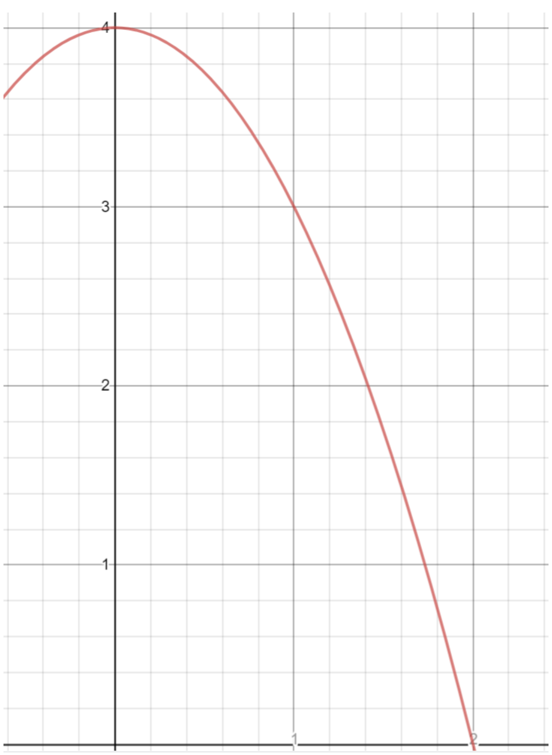
<u>Examples</u> Write the following sums using sigma notation.

a. $1 + 2 + 3 + 4 + 5$

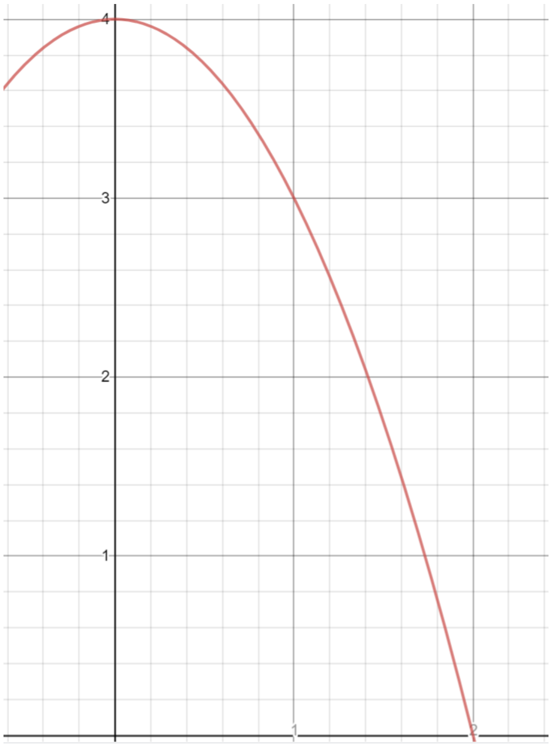
b. $2 + 4 + 6 + 8 + 10 + 12$

c. $1 - 3 + 5 - 7$

Previously we had discussed how to computer Riemann sums. Using $f(x) = 4 - x^2$ on $[0, 2]$, create 4 partitions and write out how to calculate the sum using right-hand endpoints. Can you figure out how to write this sum as a Riemann sum?



We will continue the pattern. Continuing to use $f(x) = 4 - x^2$ on $[0, 2]$, create 8 partitions and write out how to calculate the sum using right-hand endpoints. Can you figure out how to write this sum as a Riemann sum?



Can you determine how you might generalize this for n subintervals?

Write out the Riemann sum formula for using right-hand endpoints:

How can we manipulate this formula so that we use left-hand endpoints instead? Write this down:

Example Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = x^2 + 1$ on $[0,2]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.

Example: Evaluating Riemann Sums as $n \to \infty$

Example Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = 4 - x^2$ on $[0, 2]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.

Example: Evaluating Riemann Sums as $n \to \infty$

Example Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = x^3$ on $[0,1]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.

Lesson: The Basics of Definite Integrals

Before we begin, we need to define some terminology. Draw and label the following:

Let $f(x)$ be a function defined on $[a, b]$. We say that a number J is the **definite integral of f over** $[\boldsymbol{a}, \boldsymbol{b}]$ and that J is the limit of the Riemann sums $\Sigma_{k=1}^{n} f(c_k)\Delta x_k$ if the following is true:

> Given any $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for every partition P = $\{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$ we have:
>
> $|\Sigma_{k=1}^{n} f(c_k)\Delta x_k - J| < \epsilon$

This looks very similar to the definition of a limit in some ways, which I will state here:

> We have $\lim\limits_{x \to c} f(x) = L$ if, for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that:
>
> $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

However, if the limit exists and using equal width partitions, we can instead state this (label what you write down):

Write down and label the definite integral notation:

There are a few rules of definite integrals that we will want to note. When $f(x)$ and $g(x)$ are integrable functions over $[a, b]$, then we can use the following rules:

1. Order of Integration

2. Zero Width Interval

3. Constant Multiple

4. Sum and Difference

5. Additivity

6. Max-Min Inequality

7. Domination

Example if $\int_3^5 f(x)\,dx = 5$, $\int_0^3 f(x)\,dx = 2$, and $\int_0^5 g(x)\,dx = 9$, $\int_3^5 g(x)\,dx = 4$ find:

a. $\int_0^5 f(x)\ dx$

b. $\int_0^3 g(x)\ dx$

c. $\int_5^3 f(x)\ dx$

d. $\int_0^5 \big(3f(x) + g(x)\big)dx$

Finally, we have one final theorem to state that helps us determine when a function is integrable.

Theorem: Integrability of Continuous Functions

A question that usually arises from this is, “What is a function that isn’t integrable then?” This is the classic example:

Lesson: Finding Definite Integrals Using Area Formulas

If $f(x)$ is nonnegative and integrable over $[a, b]$, how do we define the area under the curve $y = f(x)$?

This means that we can calculate definite integrals using known area formulas when possible, as opposed to using Riemann sums.

Example Sketch the following curves and use known area formulas to evaluate the definite integrals.

a. $\int_{-1}^{1}(3 - |x|)\ dx$

b. $\int_{0}^{3}(2x)\ dx$

c. $\int_{0}^{2} 2x + 1\ dx$

Lesson: The Average Value of a Function

If a function $f(x)$ is integrable on $[a, b]$, then its average value or mean is calculated by:

In this lesson, we will calculate integrals only using known area formulas.

<u>Examples</u> Find the average value of the following functions:

a. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$

b. $f(x) = |x| - 1$ on $[-1, 1]$

Examples: The Fundamental Theorem of Calculus, part 2 – basic examples

Evaluate the following integrals:

a. $\int_1^4 (3x^2 - x)\ dx$

b. $\int_0^{\frac{\pi}{3}} 5\sec^2 x\ dx$

c. $\int_0^4 \left(4x^2 + \sqrt{x}\right) dx$

Examples: The Fundamental Theorem of Calculus, part 2 – tricky trig examples

Evaluate the following integrals:

a. $\int_0^{\frac{\pi}{4}} \frac{1+\cos x}{2} \, dx$

b. $\int_0^{\frac{\pi}{3}} \frac{1+\cos 2x}{2} \, dx$

c. $\int_{\frac{\pi}{2}}^{\pi} \sin^2 x \, dx$

d. $\int_0^{\frac{\pi}{2}} (\sec x + \cos x)^2 \, dx$

Examples: The Fundamental Theorem of Calculus, part 2 – tricky non-trig examples

Evaluate the following integrals:

a. $\int_3^5 (x+1)^2 \; dx$

b. $\int_1^3 \left(\frac{x^6}{3} - \frac{1}{x}\right) dx$

c. $\int_0^4 \frac{x+\sqrt{x}}{x} \; dx$

d. $\int_{-2}^2 |x| \; dx$

Examples: Which Formula Should I Use for Integrals?

In this video, we will compute the integral $\int_1^4 (2x+1)\,dx$ 3 ways:

Way 1: Area Formulas

Way 2: Riemann Sum

Way 3: Using the fundamental theorem of calculus / evaluation formula

Use the space below to write out each way.

Lesson: The Fundamental Theorem of Calculus, part 1 proof

Define the initial function we are working with and the corresponding graph:

Compute F'(x), including any relevant graphs. Interpret what this means:

State the first part of the Fundamental Theorem of Calculus

Example Find $\frac{dy}{dx}$ given $y = \int_3^x (t^5 + 1)\, dt$

Examples: The Fundamental Theorem of Calculus, part 1

Differentiate the following in 2 ways:

i. evaluate the integral then differentiate

ii. differentiate the integral directly

a. $\frac{d}{dx}\int_3^x \sin t \ dt$

b. $\frac{d}{dx}\int_1^{\cos x} 2t \ dt$

c. $\frac{d}{dx}\int_4^{\sqrt{x}} (5t^3 - \cos t) \ dt$

Examples: The Fundamental Theorem of Calculus, part 1

Compute $\frac{dy}{dx}$ for each of the following:

a. $y = \int_4^x \sqrt{3t+2}\,dt$

b. $y = \int_x^5 \sin t \; dt$

c. $y = \int_{\sin x}^4 3t^2 \; dt$

d. $y = \int_{-3}^{e^{4x}} \frac{1}{\sqrt{3t+1}} \; dt$

Step by Step Example: Fundamental Theorem of Calculus, part 1

Find $\frac{dy}{dx}$ for $y = x\int_5^{x^2} \sin t^3 \ dt$

Lesson: The Fundamental Theorem of Calculus, part 2 proof

State the Mean Value Theorem for Definite Integrals

<u>Example</u> if $\int_a^b f(x)\,dx = 0$, must there be a point c in $[a, b]$ where $f(c) = 0$?

From the Fundamental Theorem of Calculus part 1, we know that the antiderivative of f exists in the form:

$$F(x) = \int_a^x f(t)\ dt$$

Now, we know that the most general form of an antiderivative for f would be $F(x) + c$. Now we want to make this more precise.

What is $F(b) - F(a)$?

State the second part of the Fundamental Theorem of Calculus

Examples Compute the following:

a. $\int_1^3 2x\, dx$

b. $\int_0^{\frac{\pi}{2}} \cos x\, dx$

Summary: Everything about Indefinite and Definite Integrals

Compare $\int 2x\, dx$, $\int_0^3 2x\ dx$, and $\int_3^0 2x\ dx$

Antiderivatives versus Indefinite Integrals versus Differential Equations

Define an antiderivative and any special notes about it:

Define an indefinite integral and any special notes about it:

Define a differential equation and any special notes about it:

Example Find the antiderivative to the function: $2x$

Example Find the indefinite integral: $\int 2x\,dx$

Example Find the general solution to: $\frac{dy}{dx} = 2x$

Example Solve the initial value problem: $\frac{dy}{dx} = 2x; y(0) = 7$

List all of the relevant antiderivatives you know here:

Examples of Basic Antiderivatives / Indefinite Integrals

Find the antiderivatives of the following functions:

a. $\frac{1}{x^2}$

b. $3x^2$

c. $x^{-\frac{1}{2}}$

d. $\frac{2}{3}x^{-\frac{1}{3}}$

Find the indefinite integrals:

a. $\int 2\sin x \ dx$

b. $\int \sec^2 x \ dx$

c. $\int \frac{\pi}{2}\cos\frac{\pi}{2}x \, dx$

d. $\int -\csc 3x \cot 3x \ dx$

e. $\int e^{-2x} dx$

f. $\int \frac{2}{\sqrt{1-x^2}} dx$

g. $\int 3^x dx$

Examples of Basic Antiderivatives / Indefinite Integrals

Integrate the following:

a. $\int \frac{5}{x} dx$

b. $\int 5x^{\sqrt{5}} dx$

c. $\int \frac{1+\cos 3x}{2} dx$

d. $\int \cot^2 x \, dx$

e. $\int \cos x \,(\tan x + \sec x) dx$

Examples of Basic Differential Equations

Solve the initial value problems:

a. $\frac{dy}{dx} = 5 - 3x^2; y(0) = 4$

b. $\frac{dy}{dx} = 3x^{-\frac{2}{3}}; y(8) = 1$

c. $\frac{dy}{dx} = -\pi \sin \pi x\,; y(0) = 1$

Lesson: Indefinite Integrals and the Substitution Method

Before we begin, let's just review how indefinite integrals work:

$\int (3x^4 - \sin 5x + 3e^{3x})\, dx$

Now we can explain substitution using the following example: $\int (4x^3 + 2x)(x^4 + x^2)^{18}\, dx$

Example Integrate:

a. $\int (4x^3 - 6x^2)(x^4 - 2x^3)^8\, dx$

b. $\int 3\sin(3x)\ dx$

Sometimes we must manipulate the problem to get substitution to work.

Use the example $\int (6x+9)^{12}\,dx$ to demonstrate this idea:

<u>Examples</u> Integrate:

a. $\int 4\sec^2(3x+1)\ dx$

b. $\int \frac{1}{\sqrt{3x-5}}\ dx$

c. $\int \tan x\ dx$

Sometimes we must go nuclear with our manipulations. Below are a few examples of creative ways to manipulate the problems:

a. $\int \sin^5 x \cos^3 x \ dx$

b. $\int (x-4)(x+3)^{15} \, dx$

c. $\int x^3 \sqrt[3]{x^2+4} \, dx$

d. $\int \frac{dx}{1-e^x} \ dx$

Lesson: Indefinite Integrals and the Substitution Method to Solve Certain Trig Integrals

In this lesson, it is assumed that you already understand the basics of the substitution method. Here we will focus on how to use substitution to integrate the following integrals:

a. $\int \tan x \ dx$

b. $\int \cot x \ dx$

c. $\int \sec x \ dx$

d. $\int \csc x \ dx$

Basic Examples of Indefinite Integrals and the Substitution Method

a. $\int 2x\sqrt{x^2+5}\ dx$

b. $\int \frac{1}{2\sqrt{x}\,(3+\sqrt{x})^2}\ dx$

c. $\int 3\sin^3 3x \cos 3x\ \ dx$

d. $\int -\frac{1}{x^2}\sqrt{3+\frac{1}{x}}\ dx$

Examples of Indefinite Integrals and the Substitution Method Where Some Manipulation is Required

a. $\int 5x\sqrt{3x^2+2}\,dx$

b. $\int 5x^{\frac{1}{2}}\cos\left(3x^{\frac{3}{2}}+1\right)\,dx$

c. $\int \frac{\cos(5x+2)}{\sin^2(5x+2)}\,dx$

d. $\int \sqrt{\frac{x^8}{x^5+4}}\,dx$

Difficult Examples of Indefinite Integrals and the Substitution Method Where Creativity is Needed

a. $\int \sqrt{\frac{x^2-x}{x^6}}\ dx$

b. $\int 5x^5\sqrt{4x^3+1}\ dx$

c. $\int \frac{7}{4+25x^2}\, dx$

d. $\int \frac{3}{x\sqrt{1-x^4}}\, dx$

Lesson: Substitution with Definite Integrals

This lesson assumes you are already familiar with substitution for indefinite integrals. To get warmed up, evaluate the following:

a. $\int \frac{6x^2}{\sqrt{1-x^3}}\, dx$

b. $\int \frac{x}{\sqrt{1+x}}\, dx$

Using these examples, we can demonstrate 2 different ways to evaluate definite integrals with substitution.

$\int_1^3 \frac{6x^2}{\sqrt{1-x^3}}\ dx$

Way 1: Evaluate as an indefinite integral, then plug in the limits of integration

Way 2: Change the limits of integration based on your substitution

Now you try using $\int_0^3 \frac{x}{\sqrt{1+x}}\, dx$

Way 1: Evaluate as an indefinite integral, then plug in the limits of integration

Way 2: Change the limits of integration based on your substitution

Examples: Substitution with Definite Integrals

Evaluate the following:

a. $\int_1^4 \frac{1}{2x\left(1+\sqrt{x}\right)^2}\, dx$

b. $\int_0^{\frac{\pi}{3}} \cos^{-2} x \sin x \ dx$

c. $\int_1^4 x\sqrt{1+3x}\,dx$

d. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1+e^{\tan x}) \sec^2 x \; dx$

Example: Definite Integrals and Substitution

$\int_2^4 \frac{dx}{x(\ln x)^6}$

Step by Step Example: Substitution and Definite Integrals – Trig Problem!

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{x}\sin^2\left(x^{\frac{3}{2}}\right)\,dx$$

Lesson: Definite Integrals of Symmetric Functions (Proof and Examples)

In this lesson, we will look at how substitution can be used to prove the following theorem:

Theorem Suppose that $f(x)$ is a continuous function on $[-a, a]$. Then:

a. if $f(x)$ is even, then $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{2} f(x)\, dx$

b. if $f(x)$ is odd, then $\int_{-a}^{a} f(x)\, dx = 0$

Prove this below:

Examples Evaluate the following integrals:

a. $\int_{-4}^{4}(x^3 + x)\,dx$

b. $\int_{-5}^{5}(|x| - x^4 + x^2)\,dx$

Lesson: Calculating the Area Between 2 Curves With Integration

An integral is calculating what area?

Draw out an explanation of how we can understand using integration to find the area between 2 curves. Watch the explanation, and then rewrite it in terms that make sense to you.

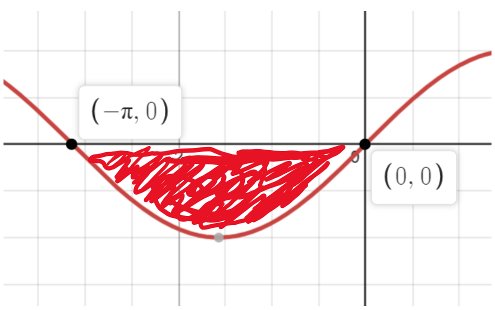
Example Find the area enclosed between $y = x^2 - 1$ and $y = 3$.

Example Find the area between $x = 2y^2$, $x = 0$, and $y = 3$

Examples: Total Area vs Area

What's the difference between area and total area? Demonstrate this concept with a graph.

<u>Example</u> Find the total area for the curve $y = \sin x$ in the shaded region (graph from Desmos.com)



Examples: The Area Between 2 Curves With Respect to x

<u>Example</u> Find the area between $y = x^2 - 2x$ and $y = -x^2 + 4x$

Examples: The Area Between 2 Curves With Respect to y

Example Find the area between $x = 2y$ and $x = y^2 - 3$

Example Find the area enclosed by $x + 4y^2 = 4$ and $x + y^4 = 1$ for $x \geq 0$

Examples: Finding Area Between Hard to Visualize Curves

a. Find the area between the curves enclosed $y = x^4 - 9x^2 + 1$ and $y = x^2 - 8$

b. Find the area enclosed by the curves $9x^2 + y = 9$ and $x^8 - y = 1$

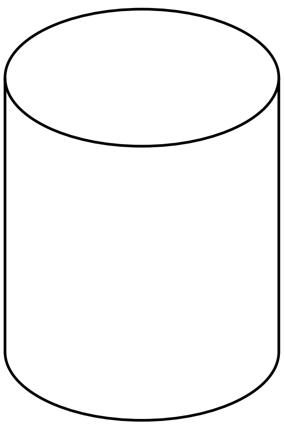
c. Find the area of the region enclosed by the curve $x = 3\sin y\ \sqrt{\cos y}$ and the line $x = 0$, $0 \le y \le \frac{\pi}{2}$

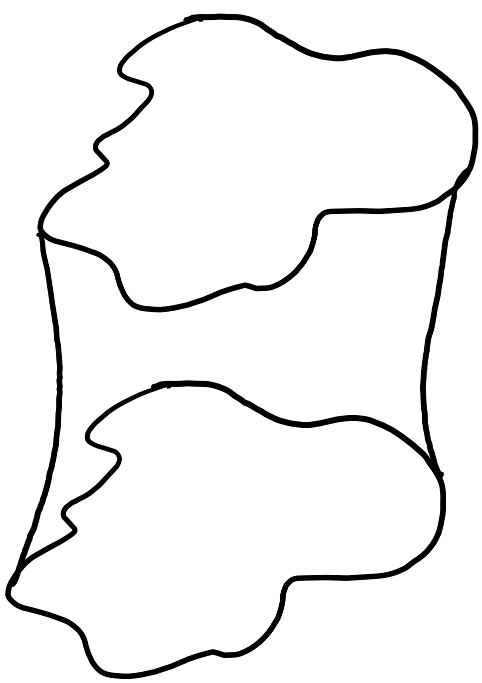
Lesson: Understanding Volume by Slicing with Parallel Planes

We’re going to discuss how integration relates to volume. One way is using cross-sections.

What is a cross-section?

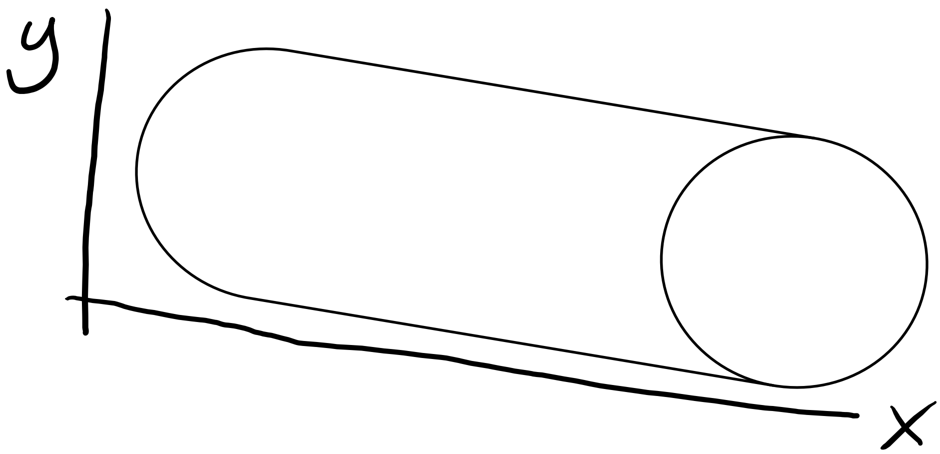
Below we have a few different kinds of cylinders. What would the cross-sections look like?





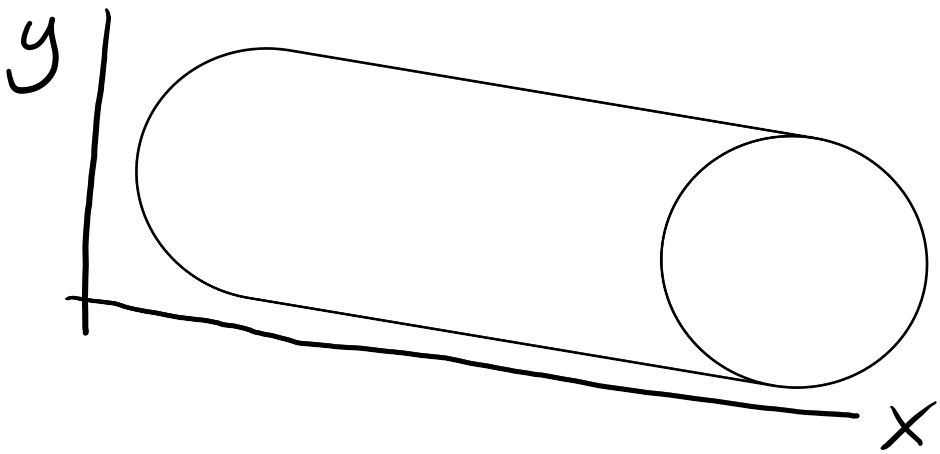
Which way do you want these slices to be oriented?

Below is how we would orient the circular cylinder.

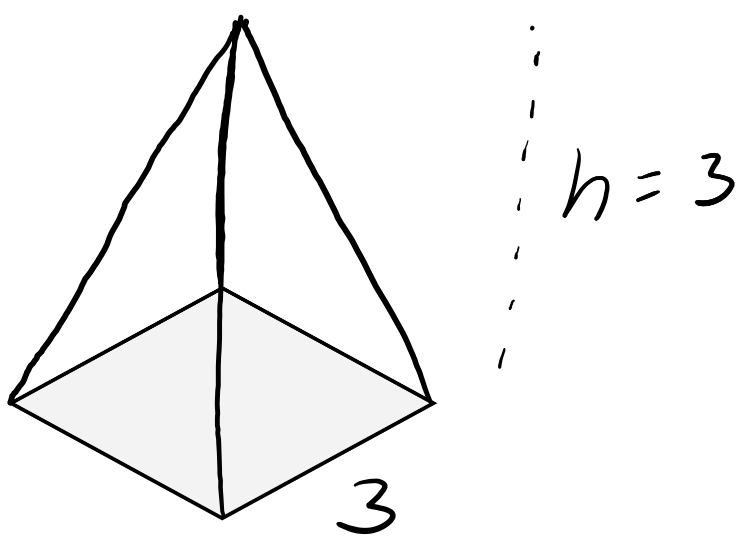


Draw a sketch showing how we would orient the second (odd looking) cylinder on the x- and y-axis.

Explain how we derive volume using cross-sections for the cylinder.



Using the following pyramid with a square base, draw how we might opt to orient this object to make it easier to analyze the cross-sections.



Summarize the explanation of how we would opt to take the volume of cross-sections in the case of the pyramid. In your explanation, explain how Riemann sums fit into this.

Write down both formulas for finding volume with cross-sections.

What are the steps to calculating the volume of a solid with cross-sections?

Example Find the volume of a solid where the base is the region between the curve $y = 4\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis, where the cross-sections are equilateral triangles with the bases running from the x-axis to the curve y.

Examples: Volumes Using Cross-Sections

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 4 - x^2$. Find the volume.

The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x$. Find the volume when the cross-sections perpendicular to the x-axis are:

a. isosceles triangles of height 4

b. semicircles with diameters running across the base of the solid

Examples: Volume by Slicing, Harder Examples

A square of side length 2x lies in a plane perpendicular to a line L. One vertex of the square lies on L. As the square moves a distance h along L, the square turns one revolution about L to generate a corkscrew shaped column with square cross sections. What is the volume?

Find the volume of the tetrahedron. Please refer to the video for the reference picture.

The base of the solid is the disk $x^2 + y^2 = 4$. The cross-sections by planes perpendicular to the y-axis between $y = -2$ and $y = 2$ are isosceles right triangles with one leg in the disk. Find the volume.

Lesson: Solids of Revolution via the Disk Method

Before you do this lesson, you need to make sure you understand the lesson about volume by cross-sections!

The disk-method is basically an application of volume by cross-sections.

What is a solid of a revolution?

What are the cross-sections in this case?

How can we represent the area of these disk?

Formula for Finding Volume by Disks for Rotation About the x-axis

Formula for Finding Volume by Disks for Rotation About the y-axis

__Example__ Find the volume of the solid generated by revolving the bounded region about the x-axis:

$$y = x^3, y = 0, x = 1$$

Example Find the volume of the solid generated by revolving the bounded region about the y-axis:

$$x = \sqrt{4\sin y}, 0 \le y \le \pi, x = 0$$

Example Find the volume of the solid generated by revolving the bounded region about the line $y = 1$.

$$y = \sqrt{2x}, y = 1, x = 2$$

Examples: The Disk-Method, Revolved around the x-axis or y-axis

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = \sec x\,, y = 0, x = -\frac{\pi}{4}, x = \frac{\pi}{4}$ around the x-axis

B. $x = \frac{2}{\sqrt{y+1}}, x = 0, y = 0, y = 3$

Examples: The Disk-Method, Revolved Around a Line (not an axis)

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = 4x, y = 0, x = 1$ about the line $x = 1$

B. $y = \sqrt{x}, y = 2, x = 0$ about the line $y = 2$

Lesson: Solids of Revolution via the Washer Method

You should understand the idea of volume by cross-sections and volume by the disk method before you try this lesson.

The washer method is used with solids of revolution, but what characteristic do these solids have?

What do we need to know about the washers?

Formula for Volume by Washers for Rotation about the x-axis

Example Find the volumes of the solid generated by revolving the enclosed region about the x-axis.

$$y = x, y = 2, x = 0$$

Example Find the volumes of the solid generated by revolving the enclosed region about the y-axis.

$$x = \tan y\,, y = 0, x = 1, y \leq 0 \leq \frac{\pi}{4}$$

Examples: The Washer Method

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = 2\sqrt{x}, y = 2, x = 0$ about the x-axis

B. $y = \sec x,\ y = \sqrt{2}, -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

Examples: Volume by Disks and Washers Around Different Axes

Find the volume of the solid generated by revolving the triangular region bounded by $y = 4x, y = 0, x = 2$ about:

a. The x-axis

b. The line $y = -2$

c. The line $y = 6$

d. The y-axis

e. The line $x = 3$

f. The line $x = -3$

Examples: When to Use Subtraction with Finding the Radius

Find the volume of the solid generated by revolving the triangular region bounded by $y = 4x, x = 0, y = 4$ about:

a. The x-axis

b. The y-axis

c. The line $y = 4$

Examples: Help! I Feel Like I'm Bad At Sketching

Sketch the region bounded by:

$y = 3x, x = 2, y = 0$

a. Show the solid of a revolution obtained by revolving this region around the x-axis

b. Show the solid of a revolution obtained by revolving this region around the y-axis

c. Show the solid of a revolution obtained by revolving this region around the line $x = 2$

d. Show the solid of a revolution obtained by revolving this region around the line $y = -2$

Sketch the region bounded by:

$y = \sqrt{x}, y = 2x - 1, x = 0$

a. Show the solid of a revolution obtained by revolving this region around the y-axis

b. Show the solid of a revolution obtained by revolving this region around the line $x = 4$

c. Show the solid of a revolution obtained by revolving this region around the line $y = 3$

d. Show the solid of a revolution obtained by revolving this region around the line $y = -2$

Lesson: Volume by Cylindrical Shells

Sometimes cross-sections can be awkward.

Demonstrate why cross-sections might be awkward using the example $y = 2x - x^2$ revolved around $x = -1$. Focus on the area enclosed by the x-axis.

Another way to calculate volume is by using a cylindrical shell. Draw how a cylindrical shell is constructed. How do we compute the volume of a shell?

Now draw how a cylindrical shell might fit into your sketch. Your sketch does not have to be perfect – you just need to use it as a way to wrap your head around the idea. Draw multiple sketches if needed. Label the pieces of your sketch.

One of the keys to using cylindrical shells is what?

Explain how the volume formula will relate to integration:

Shell Formula for A Revolution Around Any Vertical Line x=L

What assumptions do we make about calculating volume in this manner?

Example Find the volume generated by revolving the bounded region about the y-axis:

$$y = x, y = -\frac{1}{2}x, x = 4$$

Example Find the volume generated by revolving the bounded region about the line $y = -1$

$$x = \frac{1}{2}y, y = 0, x = 4$$

Examples: Volume by Using Cylindrical Shells with a Piecewise Function

Let $g(x) = \begin{cases} \frac{\sin x}{x}, & 0 < x \leq \pi \\ 1, & x = 0 \end{cases}$

a. Show the $x * g(x) = \sin x,\ 0 \leq x \leq \pi$

b. Find the volume of the solid generated by revolving the region enclosed by the curve and the x-axis about the y-axis.

Examples: The Shell Method Where The Integral is Split Up

Using the region enclosed by $y = x^2$ and $y = x + 6$, use the shell method to find the volume of solids generated by revolving the regions about the given lines:

a. $x = 3$

b. $x = -4$

c. the x-axis

d. $y = -2$

Examples: Volume by Using Cylindrical Shells About Different Lines

For the region bounded by the curves:

$$y = 2x, y = 0, x = 3$$

Find the volume generated by revolving the region about:

a. The line $x = -1$

b. The line $y = -2$

Lesson: Volume by the Washer Method vs Volume by Cylindrical Shells

Compute the volume of the solid bounded by the curves:

$$y = \frac{1}{2}x + 1, y = x, x = 0$$

In the following ways:

a. Revolved about the x-axis using the washer method

b. Revolved about the y-axis using the shell method

c. Revolved about the line x=3 using the shell method

d. Revolved about the line y=4 using the washer method

Summary: How to Set Up Volume by Integration Problems

There are a few keys to setting up volume by integration problems:

1. Always draw the region **and** draw a sketch of the solid – you should always have 2 drawings
2. Decide what formula is appropriate for the situation and with respect to which variable
3. Label your solid to go with the formula
4. Fill in the formula and integrate

Let's review a few concepts.

Write out the formulas for the disk method.

Write out the formulas for the washer method.

When do you use the washer method?

How can you determine if your revolution is a solid or has a hole in it?

With the disk/washer method, when do you integrate with respect to x?

With the disk/washer method, when do you integrate with respect to y?

In what circumstances do you have to “adjust” the radius? How does this adjustment look? What does this mean about rotating about the x-axis or y-axis?

Write out the formulas for the shell method:

With the shell method, when do you integrate with respect to x?

With the shell method, when do you integrate with respect to y?

If you rotate around the follow axis, what is the radius in that situation?

a. x-axis

b. y-axis

When do you need to subtraction with finding the shell radius?

If you need to use subtraction to find the shell radius, how do you determine the radius?

In the disk, washer, or shell methods: if you are integrating with respect to x, how can you determine the limits of integration?

In the disk, washer, or shell methods: if you are integrating with respect to y, how can you determine the limits of integration?

If you struggle to sketch your solids, I have an entire video just focusing on this. *You absolutely need to be able to quickly sketch a solid, and it is worth your time to do this video if you have anxiety about doing this.*

Example Use the region bounded by the following curves:

$$y = 3x, y = 0, x = 1$$

First, sketch the region:

a. Now we will find the volume generated by revolving this region about the x-axis. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

Example Use the region bounded by the following curves:

$$y = 3x, y = 0, x = 1$$

b. Now we will find the volume generated by revolving this region about the y-axis. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

c. Now we will find the volume generated by revolving this region about the line $x = -1$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

d. Now we will find the volume generated by revolving this region about the line $x = 2$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

e. Now we will find the volume generated by revolving this region about the line $y = 5$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

f. Now we will find the volume generated by revolving this region about the line $y = -2$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

g. Now we will find the volume generated by revolving this region about the x-axis. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

h. Now we will find the volume generated by revolving this region about the y-axis. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

i. Now we will find the volume generated by revolving this region about the line $x = -1$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

j. Now we will find the volume generated by revolving this region about the line $x = 2$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

k. Now we will find the volume generated by revolving this region about the line $y = 5$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

l. Now we will find the volume generated by revolving this region about the line $y = -2$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

Lesson: Arc Length and How the Formula is Derived

Suppose we want to find the length of the curve $y = \sqrt{x-2}$ from $x = 2$ to $x = 6$. Draw a sketch of the curve.

To understand how we can use integration to find the length of a curve, we need to recall how we find the length of a line segment. Draw out a line segment from (x_1, y_1) to (x_2, y_2), and write out how we would find the formula for this line segment.

How can we apply this idea to a Riemann sum with a curve? Draw an illustration to accompany this explanation.

Show how we can take the ideas from algebra and mold them to using integration to finding the length of the curve.

State the arc length formula.

<u>Example</u> Find the length of the curve $y = 4x^{\frac{3}{2}}$ from $x = 0$ to $x = 1$

Lesson: Arc Length with Discontinuous Derivative Values

There is an issue if we try to from the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$. Explain what the issue is.

How can we resolve this issue?

What is the formula for the length of the curve $x = g(y), c \leq y \leq d$

Show how to find the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$.

Lesson: Finding the Area of a Surface of Revolution

How do you create a solid of revolution?

How to find the area of surface from revolving the graph of f(x) around the x-axis:

How to find the area of surface from revolving around the y-axis:

<u>Examples</u> Find the areas of surface generated by revolving the curve around the given access.

a. $f(x) = \sqrt{x+1}, 1 \leq x \leq 5$, x-axis

b. $x = \left(\frac{1}{3}\right)y^{\frac{3}{2}} - y^{\frac{1}{2}}, 1 \leq y \leq 3$, y-axis

Examples: Finding Surface Areas of Revolution

Find the surface area of the revolution generated by revolving the given curve about the given axis.

a. $x = \sqrt{4y - 3}, 1 \leq y \leq 3$, y-axis

b. $y = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}}, 1 \leq x \leq 2$, x-axis

c. $x = \frac{1}{2}(e^{y} + e^{-y}), 0 \leq y \leq \ln 3$, y-axis

Lesson: A Guide to Approaching Work Problems

Constant force work formula:

Variable force work formula:

A note on calculating force:

The units are very important because they will likely guide you on how to do the problem.

Let's note units too for the following:

Measure	SI Units	US Customary
Force		
Work		
Mass		
Length		
Acceleration due to gravity		

What are some important notes regarding how we think of force in different applications?

With problems involving objects:

With pumps/volume problems:

If you have completely forgotten how to do dimensional analysis, *please watch my review video on this topic.* This will really help you!

Going back to our integral, rewrite the integral to explain how we should be watching out for units.

A guideline for how to get to these units:

With objects

With pumps/volume

Example You want to haul a 25-m length of hanging rope. How much work will it take if the rope weighs 0.624N/m.

Why is it obnoxious to say “weighs” here? What’s a better way to think about this?

What elements go into how this formula is set up?

When we set up our integral, what ultimately needs to happen?

Show how to finish the problem and show how the units guide us that we are right.

Example You want to haul a 25-m length of hanging rope. How much work will it take if the rope weighs 0.624N/m.

What if I wanted to use (25-x) instead of x to model the length left of the rope?

What does this mean?

Note the following fact about integration:

Example You want to haul a 15-m length of hanging rope. How much work will it take if the rope weighs 0.08 kg/m.

Example A bag of sand weighing 160lbs was lifted at a constant rate. As it rose, sand leaked out at a constant rate. One-quarter of the sand was gone when the bag had been lifted to 20ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment).

Example A vertical right-circular cylindrical tank measures 20 ft high and 10 ft in diameter. It is full of milk weighting 64 $\frac{lb}{ft^3}$. How much work does it take to pump the milk to the level of the top of the tank?

Example Suppose a 10ft tall, 10ft diameter conical tank (tip down) is filled 5 ft high with kerosene weighing 51.2 $\frac{lb}{ft^3}$. How much work does it take to pump the kerosene to the rim of the tank? (See picture)

Example Suppose a 10ft tall, 10ft diameter conical tank (tip down) is filled 8ft high with kerosene weighing 51.2 $\frac{lb}{ft^3}$. How much work does it take to pump the kerosene to 5ft above the top of the tank?

Lesson: Hooke's Law for Springs

What is Hooke's law?

What are the relevant units to pay attention to?

<u>Example</u> It took 1500 J of work to stretch a spring from its natural length of 2m to a length of 4m. Find the spring's force constant.

Example If a force of 80 N stretch a spring 2 m beyond its natural length, how much work does it take to stretch the spring 6 m beyond its natural length?

Example A spring has a natural length of 8 in. A 500-lb force stretches the spring to 13 in.

a. Find the force constant

b. How much work is done in stretching the spring from 8 in to 13 in?

c. How far beyond its natural length will a 800-lb force stretch the spring?

Lesson: Fluid Forces

Fluid Force on a Constant-Depth Surface

Fluid Force Against a Vertical Flat Plate (Integration Formula)

<u>Example</u> Refer to the video for the sketch. Calculate the fluid force on one of the triangular plate using the given coordinates.

Example In a pool fill with water to a depth of 12 ft, calculate the fluid force on one side of a 4ft x 6ft rectangular plate if the plate rests vertically at the bottom of the pool on its 4-ft edge.

Examples: Fluid Forces

Calculate the fluid force on one side of a semicircular plate of radius 4 ft that rests vertically on its diameter at the bottom of a pool filled with water to a depth of 7 feet.

A metal cube that is 8ft on each side has a parabolic gate on one side at the bottom. See the picture in the video. It is designed to withstand a fluid force 500 lb without rupturing. The liquid you plan to store has a weight-density of 62.4 lb/ft^3.

a. What is the fluid force on the gate when the liquid is 6 feet deep?

b. What is the maximum height to which the container can be filled without exceeding the gate's design limitation?

Examples: The Logarithm Defined as an Integral

Evaluate the following integrals.

a. $\int_{-4}^{-2} \frac{dx}{x}$

b. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

c. $\int 3x^2 e^{-x^3} dx$

d. $\int_{\ln\left(\frac{\pi}{3}\right)}^{\ln\left(\frac{\pi}{2}\right)} 2e^x \sin e^x \, dx$

e. $\int_1^3 x^{3x}(1 + \ln x)dx$

f. $\int_1^9 \frac{\log_3 x}{x} dx$

Examples: The Logarithm Defined as an Integral, Initial Value Problems

Solve the initial value problems.

a. $\frac{dy}{dx} = e^{-x}\csc^2(\pi e^{-x}),\ y(\ln 4) = \frac{2}{\pi}$

b. $\frac{d^2y}{dx^2} = 3e^{-x}, y(0) = 1, y'(0) = 0$

c. $\frac{d^2y}{dx^2} = \sec^2 x,\ y(0) = 0, y'(0) = 1$

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Examples: Separable Differential Equations, General Solutions Only

<u>Examples</u> Find general solutions to the following:

a. $\frac{dy}{dx} = 2x\sqrt{y}$

b. $\frac{dy}{dx} = (xy)^{\frac{3}{2}}$

c. $\frac{dy}{dx} = \frac{(x^5-1)y^3}{x^2(3y^2-2)}$

Examples: Separable Differential Equations, Initial Value Problems

Examples Solve the initial value problems

a. $\frac{dy}{dx} = \sqrt{y}; y(0) = 4$

b. $\frac{dy}{dx} = \frac{x}{y}; y(2) = 6$

c. $\frac{dy}{dx} = 3x^2y^2 - y^2; y(0) = \frac{1}{2}$

Examples: Population Growth

We can study how diseases die when properly treated by using the following logic let $\frac{dy}{dt}$ represent the rate at which the number of infected people changes is proportional to the number y. The number of people who are cured is proportional to y, the number who are infected with the disease. As of April 23, 2020, the number of coronavirus cases was 2,659,557. For simplicity, let's round that number to 2,660,000. Let's suppose that we can reduce the number of cases of the disease by 50% in a year. If there are 2,660,00 cases, then how many years will it take to the reduce the number to 2,000 cases?

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Examples: Radioactivity and Exponential Decay

Some atoms are unstable and spontaneously emit mass, which is called radioactive decay. An element that goes through this process is called radioactive. The decay of a radioactive element is described by the differential equation:

$$\frac{dy}{dy} = -ky, k > 0$$

If y_0 is the number of initial nuclei present, then the number at a later time is:

$$y = y_0 e^{-kt}, k > 0$$

An interesting consequence of this is that half-life can be calculated by the equation:

$$\frac{\ln 2}{k}$$

The half-life of the plutonium isotope is 24,360 years. If 15 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 75% of the isotope to decay?

Examples: Newton's Law of Cooling

A metal beam was brought from the outside cold into a machine shop where the temperature was held at $75°\mathrm{F}$. After 10 min, the beam warmed to $45°\mathrm{F}$ and after another 15 min it was $55°\mathrm{F}$. Use Newton's Law of Cooling to estimate the beam's initial temperature.

Lesson: Intro to Hyperbolic Functions

The hyperbolic sin and hyperbolic cosine functions are defined by the equations:

How do we say “sinh x” ?

What about “cosh x” ?

Using these definitions, we get some interesting identities. Determine the value of $2\sinh x \ \cosh x$:

Now determine the value of $\cosh^2 x - \sinh^2 x$

List all of the relevant identities for hyperbolic functions:

Lesson: Derivatives of Hyperbolic Functions – A Quick Overview

Find the derivative of $\sinh x$

Find the derivative of $\operatorname{csch} x$

Find the derivative of $\tanh x$

Write the derivative rules for all of the hyperbolic functions below:

Write the integral formulas for all of the hyperbolic functions:

Lesson: Integration by Parts

State the formula for **integration by parts**:

We will look at a few basic examples in this video.

Basic Example $\int x \cos \pi x \; dx$

<u>Basic Example</u> $\int \ln x \ dx$

<u>Example of Using Integration by Parts Multiple Times</u> $\int x^3 e^x \ dx$

Example Where the Integral Cycles $\int e^x \sin x \ dx$

Lesson: Integration by Parts with Definite Integrals

Let's look at an example of how to use integration by parts with definite integrals.

$\int_0^4 x\, e^{-x}\ dx$

Examples: Integration by Parts, Examples Where You Have to Use Integrate Multiple Times

a. $\int x^2 \cos x \ dx$

b. $\int (x^2 - 2x + 1)e^x \, dx$

Examples: Integration by Parts, Examples Where The Integral Cycles Back to Itself

a. $\int e^{4x} \cos 3x \; dx$

b. $\int e^{-x} \cos x \, dx$

Examples: Using Substitution AND Integration by Parts Together

a. $\int e^{\sqrt{4x+1}}dx$

b. $\int x\,(\ln x)^2\,dx$

c. $\int \cos \sqrt{x}\, dx$

Lesson: Trigonometric Integrals with Different Powers of Sine and Cosine

I. Integrals of the Form $\int \boldsymbol{sin^m x\, cos^n x\ \ dx}$

A. When m is even, what should you do?

<u>Example</u> $\int \sin^3 4x \cos^2 4x\ \ dx$

B. When n is odd, what should you do?

Example $\int \cos^3 2x \; dx$

C. When both m and n are odd, what should you do?

Example $\int \sin^4 x \cos^2 x \; dx$

Examples: Integrals with Powers of Sines and Cosines

a. $\int \sin^4 3x \cos 3x \ \ dx$

b. $\int \cos^3 4x \sin^5 4x \ \ dx$

c. $\int 4\, sin^4\, 2\pi x \ \ dx$

Lesson: Trigonometric Integrals with Square Roots or Different Products of Sine and Cosine

II. Integrals with Square Roots

When an integral has a square root with sine or cosine, what identity should we try to leverage?

<u>Example</u> $\int_0^{\frac{\pi}{2}} \sqrt{1-\cos 2x}\ dx$

Example $\int_0^{\frac{\pi}{2}}(1-\cos^2 x)\,dx$

Example $\int_0^{\frac{\pi}{6}}\sqrt{1+\sin x}\,dx$

III. Integrals with Different Products of Sine and Cosine

For integrals of the form:

- $\int \sin mx \sin nx \, dx$
- $\int \cos mx \cos nx \, dx$
- $\int \sin mx \cos nx \, dx$

What trig identities can be used to help integrate these types of integrals?

Example $\int \sin 7x \cos 3x \; dx$

Examples: Integrals and Square Roots with Trig Functions

a. $\int_0^{\pi} \sqrt{1-\cos^2 x}\ dx$

b. $\int_0^{\frac{\pi}{2}} x\sqrt{1-\cos 2x}\, dx$

Examples: Integrals and Products of Sines and Cosines

a. $\int \sin 3x \cos 2x \, dx$

b. $\int \cos^2 2x \sin x \ \ dx$

Lesson: Trigonometric Integrals with Tangent, Cotangent, Secant, and Cosecant

For integrals with $\tan x$ and $\sec x$, what identities should be used to help with integration?

For integrals with $\cot x$ and $\csc x$, what identities should be used to help with integration?

<u>Examples</u> Integrate the following.

a. $\int \tan^3 x \ dx$

b. $\int \sec^4 x \ dx$

c. $\int \cot^2 x \csc^4 x \, dx$

Examples: Integrals and Powers of Tangent and Secant

a. $\int \sec^2 x \tan^2 x \ dx$

b. $\int \sec^3 x \tan x \ dx$

c. $\int \csc^4 x \ dx$

Lesson: Trigonometric Substitution

Trig substitution is a special way to use substitution for integrals of a special form.

Reference Guide

If the integral involves…	Then substitute:	And use:

<u>Example</u> Write out instructions for how to approach these types of problems: $\int \frac{dx}{\sqrt{9+x^2}}$

What is another way to integrate $\int \frac{dx}{\sqrt{9+x^2}}$?

$\sinh^{-1}\frac{x}{a}$ is equivalent to:

<u>Example</u> Evaluate $\int \frac{x^2}{\sqrt{25-x^2}}dx$

Example $\int_0^2 \frac{dx}{50+2x^2}$

Example Use substitution before using trig substitution: $\int_{\frac{1}{12}}^{\frac{1}{4}} \frac{3}{\sqrt{x}+4x\sqrt{x}}\, dx$

Examples: Basic Trig Substitution Examples

a. $\int \frac{dx}{\sqrt{25+x^2}}$

b. $\int \sqrt{16-x^2}dx$

Examples: More Complicated Trig Substitution Examples

a. $\int \sqrt{\frac{x+1}{1-x}}\,dx$

b. $\int \frac{dx}{(x^2-4)^{\frac{3}{2}}}$

c. $\int \frac{6}{(4x^2+1)^2}\,dx$

Examples: Integrals That Use Both Substitution and Trig Sub

$\int \frac{e^x}{1+e^{2x}}\, dx$

Review: How to Complete the Square (A Few Ways)

Complete the square on the polynomial:

a. $x^2 - 4x$

b. $x^2 + 6x + 9$

c. $x^2 + 3x$

d. $5x^2 - 20x - 10$

e. $6 - 2x - x^2$

Review: Completing the Square for Advanced Integration Techinques

First, let's review a few examples of completing the square in general. Complete the square on the following:

a. $5x^2 + 10x - 20$

b. $4 - 2x - x^2$

c. $x^2 + 3x - 7$

Now that we have thoroughly reviewed this skill, let's use it for 2 different integrals.

Integrate the following.

a. $\int \frac{dx}{\sqrt{2x+x^2}}$

b. $\int \sqrt{5-4x-x^2}\ dx$

Lesson: Integration by Partial Fractions

What are the keys to writing an expression $\frac{f(x)}{g(x)}$ with partial fractions?

I. Nonrepeated Linear Factors

$\int \frac{3x+11}{x^2-x-6}\,dx$

II. Repeated Linear Factors

$\int \frac{3x-2}{x^2-2x+1} dx$

III. Irreducible Quadratic Factors

$\int \frac{x}{x^4-x^2-2}\,dx$

IV. Improper Fractions

$\int \frac{x^3+1}{x^3-x}\, dx$

Summary of How to Solve Partial Fractions

Factor in Denominator	Form for Partial Fractions
$ax + b$ (unique factor)	
$(ax + b)^k$	
$ax^2 + bx + c$	
$(ax^2 + bx + c)^k$	

Examples: Integration with Partial Fractions Where the Denominator Has Linear Factors Only

a. $\int \frac{dx}{x^3+x^2-2x}$

b. $\int \frac{x^2}{(x-1)(x^2+2x+1)} dx$

c. $\int \frac{2x+1}{x^2-7x+12}\, dx$

Examples: Integration with Partial Fractions Where the Denominator Has Quadratic Factors

a. $\int \frac{x^2-x+2}{x^3-1}$

b. $\int \frac{2x^3+5x^2+8x+4}{(x^2+2x+2)^2}$

Examples: Integration with Partial Fractions With an Improper Fractions

$$\int \frac{16x^3}{4x^2 - 4x + 1}\ dx$$

Examples: Some Hard Integrals that Involve Partial Fractions and Creativity

a. $\int \frac{\cos x}{\sin^2 x - \sin x - 6} dx$

b. $\int \frac{1}{x^{\frac{3}{2}} - x^{\frac{1}{2}}} dx$

c. $\int \frac{1}{\cos x+\sin 2x}\, dx$

Lesson: A Review of "Basic" and Creative Integration Techniques

Write down all of the "basic" integration formulas we should know at this point:

Now we will review some more creative "basic" integration techniques.

I. More Creative Substitution Ideas with Completing the Square

$\int \frac{dx}{\sqrt{2x-x^2}}$

II. Manipulating an Expression to Use Substitution

$\int \frac{dx}{\sqrt{e^{2x}-1}}$

III. Multiply by the Conjugate

$\int \frac{dx}{1-\sec x}$

IV. Multiply by an Expression Equivalent to 1

$\int \frac{1}{e^x+e^{-x}}\, dx$

V. Split Into 2 Parts

$\int \frac{1+\sin x}{\cos^2 x}\ dx$

VI. Use Long Division

$\int \frac{4x^2-7}{2x+1} dx$

Lesson: Completing the Square to Integrate

$\int \frac{dx}{\sqrt{2x-x^2}}$

Lesson: Multiplying by a Strategic Fraction to Integrate

a. $\int \frac{dx}{1-\sin x}$

b. $\int \frac{dx}{\sec x+\tan x}$

Lesson: Improper Fractions and Division

What makes an improper fraction?

a. $\int \frac{x^2}{x^2+1} dx$

b. $\int \frac{4x^3-x^2+16x}{x^2+4}\ dx$

Lesson: Separating Fractions to Integrate

$\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

Lesson: Using Substitution More Creatively

a. $\int \frac{dx}{\cos^2 x \tan x}$

b. $\int \frac{2^{\ln x^3}}{16x}\, dx$

Lesson: The Trapezoidal Rule

With Riemann sums, we use rectangles for making area estimates. However, there are other shapes we can use.

First we need to know the area of a trapezoid formula:

$$A = \frac{a+b}{2}h$$

You may want to draw a reference picture for that.

Now we can discuss how to use trapezoids for area estimates below:

State the Trapezoidal Rule

<u>Example</u> Estimate the area of $f(x) = 2x + 1$ on $[0, 2]$ using four subintervals and the trapezoidal rule.

Lesson: Improper Integrals Type I – Integrals Containing Infinity

In this lesson we will consider type 1 improper integrals. What are some examples of these types of integrals?

We will use limits to help us integrate these types of integrals. Explain the different ways to solve integrals of this type:

If the integral has an answer, what do we say about that integral?

What if the integral does not have a finite answer?

Find the area under the curve $y = \frac{1}{x^2+1}$ from $x = 0$ to $x = \infty$ if possible.

Example $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Lesson: Improper Integrals Type II – Integrals Involving Vertical Asymptote

Explain what type of integrals are type II integrals:

How do you evaluate these types of integrals?

Show how to integrate $\int_0^3 \frac{dx}{\sqrt{3-x}}$

Example $\int_{-8}^{1} \frac{1}{x^{\frac{1}{3}}} \, dx$

Lesson: Comparison Tests for Integrals

Why might we want to use a comparison test for an integral?

Explain the Direct Comparison Test

Examples Use the direct comparison to evaluate the following integrals.

a. $\int_1^{\infty} e^{-x^2}\,dx$

b. $\int_1^{\infty} \frac{1}{\sqrt{x^2-0.01}}\,dx$

c. $\int_0^{\pi} \frac{dx}{\sqrt{x}+\sin x}$

Explain the Limit Comparison Test

Example Use the limit comparison test to evaluate the integrals.

a. $\int_1^\infty \frac{1}{1+x^2}\, dx$

b. $\int_0^1 \frac{dx}{x-\sin x}$

Summary: L'Hopital's Rule

State L'Hopital's Rule

What are all the indeterminate forms that apply to L'Hopital's Rule?

For which type of indeterminate forms can you just take the derivative of the top and bottom?

<u>Examples</u> Find the following limits.

a. $\lim_{x\to 0} \frac{4x^2}{2\cos x - 2}$

b. $\lim_{x\to \infty} \frac{\ln x}{3\sqrt[3]{x}}$

For the forms $0 * \infty$ or $\infty - \infty$, what must you try?

Examples Find the following limits.

a. $\lim_{x\to\infty} 3x\, e^{-x}$

b. $\lim_{x\to 0^+} (\ln 5x - \ln(\sin x))$

What must you do with indeterminate powers?

Example Find the limit $\lim_{x\to 0^+} x^{\frac{-1}{\ln x}}$

What happens if L'hopitals rule continues to cycle?

Example Find the limit $\lim_{x\to 0^+} \frac{\cot x}{\csc x}$

Summary: How to Study for the Integration Exam

Obviously you are told over and over again that you need to practice for an integration exam. If you go through each section of your notes / book / homework, what techniques are used?

You should be able to do a few problems of each of these types. In this video, we will take a look at one type of common integral form that can sprawl into many different types of problems.

Analyze the following, determine appropriate techniques for integration, and then integrate.

a. $\int \sqrt{4-x^2}\,dx$

b. $\int_0^1 \frac{5x+5}{\sqrt{x^2+2x}}\,dx$

c. $\int \frac{dx}{x^2+4x}$

d. $\int \frac{dx}{\sqrt{x^2-2x}}$

e. $\int \frac{x^3}{x^2+2x+1}\,dx$

f. $\int \frac{1}{\sqrt{x^2-4x+5}}\,dx$

a. $\int \sqrt{4-x^2}\,dx$

b. $\int_0^1 \frac{5x+5}{\sqrt{x^2+2x}}\,dx$

c. $\int \frac{dx}{x^2+4x}$

d. $\int \frac{dx}{\sqrt{x^2-2x}}$

e. $\int \frac{x^3}{x^2+2x+1}\,dx$

f. $\int \frac{1}{\sqrt{x^2-4x+5}}\,dx$

Lesson: Introduction to Sequences

What is a sequence?

What is the index of a sequence?

How do we describe the length of a sequence?

Write does a few examples of how sequences can be presented:

Example Write down the first 5 terms of each sequence.

a. $a_n = \frac{1}{n!}$

b. $a_n = 2 + (-1)^n$

What is a recursive sequence?

Find the first six terms of the recursive sequence if:

$a_1 = 2, a_n = \frac{na_{n-1}}{n-1}$

Lesson: Introduction to Convergence of Sequences

When does a sequence converge?

What a sequence does not converge, what do we say about it?

What notation do we use to denote a sequence has converged?

How do we define sequence divergence?

Example Use the definition of sequence convergence to prove that $\lim_{n\to\infty} \frac{1}{n} = 0$

Example Use the definition of sequence convergence to prove that $\lim_{n\to\infty} c = c$ for any constant c.

Rules for Limits of Sequences

Let a_n and b_n be sequences. Let $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then the following rules hold:

Example Determine if the following converge or diverge. If they converge, then what is the limit?

a. $a_n = \frac{1-3n}{1+3n}$

b. $a_n = \sqrt{n}$

Example Determine if the following converge or diverge. If they converge, then what is the limit?

a. $a_n = \frac{3+(-1)^n}{n}$

b. $a_n = \sqrt{\frac{2n}{n+1}}$

Explain the Sandwich Theorem for Sequences

Example Determine if and where the sequence converge to: $a_n = \frac{\sin n}{n}$

State the Continuous Function Theorem for Sequences

Example Determine if and where the sequence converge to: $a_n = \sqrt{\frac{n+1}{n}}$

Explain the theorem that connects functions to sequences:

What is the cool result of this theorem?

Example Explain why $\lim_{n\to\infty} \frac{\ln n}{n^2} = 0$

Example Determine if and where the sequence converge to: $a_n = \left(\frac{4n+1}{4n-1}\right)^n$

Lesson: Commonly Occurring Limits for Sequences

State the 6 commonly occurring limits for sequences:

Example Determine whether the following sequences converge and to what:

a. $a_n = \frac{(-5)^n}{n!}$

b. $a_n = \sqrt[n]{5^{2n+1}}$

<u>Example</u> Determine whether the following sequences converge and to what:

a. $a_n = \frac{\ln n}{n^{\frac{1}{n}}}$

b. $a_n = \left(1 - \frac{1}{n}\right)^n$

c. $a_n = \frac{7^n}{n!}$

Lesson: Boundedness and Monotonicity for Sequences

Explain the concepts of being bounded from above versus bounded from below:

What does it mean to be a bounded versus an unbounded sequence?

Example Describe a sequence with a lower bound but no upper bound:

Example Describe a sequence with an upper bound but no lower bound:

Example Explain a bounded sequence that does not converge:

Explain the difference between nonincreasing vs nondecreasing:

What does it mean for a sequence to be monotonic?

Example Write an example of each type of sequence:

a. A nondecreasing sequence

b. A nonincreasing sequence:

c. A sequence that is not monotonic:

State the Monotonic Sequence Theorem:

Examples: Determining the Convergence of Sequences

Determine if the following sequences converge or diverge.

a. $a_n = \frac{3^n}{n^3}$

b. $a_n = \frac{(2n+2)!}{(2n-1)!}$

c. $a_n = (\ln n - \ln(n-1))$

d. $a_n = \tan^{-1} n$

Examples: Determining the Convergence of Recursively Defined Sequences

Determine if the following sequences converge or diverge.

a. $a_1 = -1, a_{n+1} = \frac{a_n+6}{a_n+2}$

b. $a_1 = 3, a_{n+1} = 8 - \sqrt{a_n}$

Lesson: Intro to Infinite Series

Write down an example of a sequence:

Using that sequence, show how we can turn a sequence into a series:

What is the difference between a finite series and an infinite series?

Show how we can denote summing up only n terms:

In some cases, a pattern can emerge with infinite series. Let's investigate the sum of $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots$

Partial Sum	Value of Sum	Another way to think of this
$S_1 = 1$	1	
$S_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$	
$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$	
$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$	

How do we denote a sequence of partial sums?

How do we denote an infinite series?

What happens if the partial sums of an infinite series converge to a limit L?

Use the sum $a_n = 1 + \frac{1}{n}$ show how we can visualize the sums of a series with rectangles:

Use the sum $a_n = \frac{1}{n^2}$ show how we can visualize the sums of a series with rectangles:

Lesson: Geometric Series

What is the general form of a geometric series?

Show an example of a geometric series:

What's are two ways to write a geometric series?

What is the ratio of a geometric series?

True or false: The ratio of a geometric series can only be positive

Explain how we can evaluate the value of a geometric series.

Explain when a geometric series converges versus diverges:

Example Write out the first 4 terms of the series, then determine if the series converges or diverges.

$$\sum_{n=0}^{\infty} (\frac{(-1)^n}{5^n}$$

Example Determine if the geometric series converges or diverges. If it converges, find its sum.

$$\frac{9}{4} - \frac{27}{8} + \frac{81}{16} - \frac{243}{32} + \cdots$$

If $\Sigma a_n = A$ and $\Sigma b_n = B$ are convergent series, then what other properties hold?

Example Write out the first 4 terms of the series, then determine if the series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$$

Example Express the number as the ratio of two integers: $0.\overline{231} = 0.231\ 231\ 231\ 231\ \ldots$

Lesson: Nth Term Test for a Divergent Series

If $\Sigma_{n=1}^{\infty} a_n$ converges, what can be said about a_n?

What is the nth-Term Test for Divergence

<u>**Examples**</u> Why do the following series diverge?

a. $\Sigma_{n=1}^{\infty} n$

b. $\Sigma_{n=1}^{\infty} \left(\frac{2n+3}{n}\right)$

c. $\Sigma_{n=1}^{\infty} (-1)^{n+1}$

Lesson: Combing Series, Reindexing Terms in a Series, and Adding or Deleting Terms in a Series

How do you find the value of the series $\sum_{n=5}^{\infty}\left(\frac{1}{4^n}\right)$? Note the index.

How can we reindex the series $\sum_{n=1}^{\infty}\left(\frac{1}{7^{n-1}}\right)$?

Examples: Working with Geometric Series

Determine if the following series converge or diverge.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}$

b. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

Examples: Repeating Decimals and Geometric Series

Express each number as the ratio of two integers.

a. $0.\overline{27} = 0.27\ 27\ 27\ 27\ ....$

b. $1.35\overline{127} = 1.35\ 127\ 127\ 127\ ...$

Examples: The nth Term Test for Divergence

Use the nth-term test for divergence to show that the series is divergent OR that the test is inconclusive.

a. $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$

b. $\sum_{n=0}^{\infty} \frac{n}{n^2+7}$

Examples: Partial Sums with Telescoping Series

Find a formula for the nth partial sum, then use it to determine if the series converges or diverges.

a. $\sum_{n=1}^{\infty}(\tan n - \tan(n-1))$

b. $\sum_{n=1}^{\infty}\frac{6}{(2n-1)(2n+1)}$

Lesson: The Integral Test for Infinite Series

When does a series of nonnegative terms converge?

Explain how this idea applies to the harmonic series $\Sigma_{n=1}^{\infty} \frac{1}{n}$

State the Integral Test

Example Does the following series converge: $\Sigma_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$

For what values of p does the p-series converge?

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

Example Determine if the following series converge by the Integral Test

a. $\sum_{n=1}^{\infty}\left(\frac{n}{n^2+5}\right)$

b. $\sum_{n=2}^{\infty}\left(\frac{1}{n(\ln n)^2}\right)$

Examples: Using the Integral Test for Series

Determine which series converge or diverge using the integral test.

a. $\sum_{n=1}^{\infty} e^{-2n}$

b. $\sum_{n=1}^{\infty} \frac{n^2}{e^{\frac{1}{3}n}}$

Lesson: Comparison Tests for Series

State the direct comparison test:

Example Show how to use the direct comparison test to determine if the series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$$

State the Limit Comparison Test:

<u>Example</u> Use the limit comparison test to determine if the series converges or diverges:

$$\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^{n}$$

Example Use the limit comparison test to determine if the series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

Examples: Comparison Tests for Series

Use any methods to determine if the following converge or diverge.

a. $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

b. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

c. $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$

d. $\sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{2.3}}$

Lesson: The Alternating Series Test

State the Alternating Series Test

<u>Example</u> Does the alternating harmonic series converge? $\Sigma_{n=1}^{\infty}\left((-1)^{n+1}\left(\frac{1}{n}\right)\right)$

Example Determine if the alternating series converge or diverge. Not all series satisfy the conditions of the Alternating Series Test.

a. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n+1}1}{n5^n}\right)$

b. $\Sigma_{n=1}^{\infty}((-1)^{n+1}\left(\frac{n}{5}\right)^n$

c. $\Sigma_{n=1}^{\infty}\left((-1)^n(\frac{n}{n^2+1})\right)$

Lesson: Conditional Convergence

Explain what conditional convergence is:

Explain how conditional convergence applies to the alternating p-series: $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{n^p}\right)$

What is the Rearrangement Theorem for Absolutely Convergent Series?

<u>Example</u> What happens if we multiply the alternating harmonic series by 2?

Lesson: Summary of Tests to Determine Convergence or Divergence

State the 6 different tests we've discussed so far for infinite series.

Example Using each test at most once, determine the convergence of the following series.

a. $\sum_{n=1}^{\infty} \left(\frac{(-1)^n n}{n+5}\right)$

b. $\sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}(0.5)^n}{n}\right)$

c. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\sqrt{n}}\right)$

d. $\Sigma_{n=1}^{\infty}(-6)^{-n}$

e. $\Sigma_{n=1}^{\infty}\left(\frac{(-105)^n}{n!}\right)$

f. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n+1}(\sqrt{n}+1)}{n+1}\right)$

Examples: Alternating Series Test

Determine if the following satisfy the conditions of the alternating series test, and then determine if the series converges.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{n^2+5}{n^2+4}$

b. $\sum_{n=1}^{\infty}(-1)^n \ln\left(1+\frac{1}{n}\right)$

Examples: Conditional Convergence

Determine if the series converges absolutely, if it converges, or if it diverges.

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+2}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+3}$

c. $\sum_{n=1}^{\infty}(-1)^{n+1}\sqrt[n]{13}$

d. $\sum_{n=1}^{\infty}(-1)^n \frac{(-2)^{n+1}}{n+5^n}$

e. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \tan^{-1} n}{n^2+1}$

Lesson: Absolute Convergence for Infinite Series

What does it mean for a series to converge absolutely?

What is the Absolute Convergence Test?

Examples Determine if the following series converge or diverge absolutely.

a. $\Sigma_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^3}\right)$

b. $\Sigma_{n=1}^{\infty}\left(\frac{\cos n}{n^2}\right)$

Lesson: The Ratio Test

State the Ratio Test

Examples Use the Ratio Test to determine the convergence of the series.

a. $\sum_{n=1}^{\infty}\left(\frac{(3n)!}{n!n!}\right)$

b. $\sum_{n=1}^{\infty}\left(\frac{n!n!}{(3n)!}\right)$

Lesson: The Root Test

What is the Root Test?

Example Use the Root Test to determine if the series converge absolutely or diverge.

a. $\sum_{n=1}^{\infty}\left(\frac{5n-3}{4n+7}\right)^n$

b. $\sum_{n=1}^{\infty}\left(\sin^n\left(\frac{1}{\sqrt[3]{n}}\right)\right)$

Examples: The Ratio Test

Determine if the series converges absolutely or if it diverges.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{n+5}{5^n}$

b. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+4)^2}$

Review: Factorials within a Series

Back to basics: use $f(x) = x^2 - 2x + 1$ to compute the following

a. $f(3)$

b. $f(n+1)$

Now let's compute $\frac{u_{n+1}}{u_n}$ for the following:

a. $\frac{3^n}{(n+1)!}$

b. $\frac{(2n)!}{(3n+1)!}$

Compute the ratio test for the following:

$$\sum_{n=1}^{\infty}\left(\frac{(n!)^2}{(2n)!}\right)$$

Examples: The Root Test

Determine if the series converges absolutely or if it diverges.

a. $\sum_{n=1}^{\infty} \frac{5^n}{(4n)^n}$

b. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)^{n^2}$

Examples: Series with Recursively Defined Terms

Determine if the series $\sum_{n=1}^{\infty} a_n$ converge or diverge.

a. $a_1 = 2\ a_{n+1} = \frac{1+\cos n}{n}\ a_n$

b. $a_1 = 5, a_{n+1} = \frac{\sqrt[n]{n}}{2}\ a_n$

Lesson: Intro to Power Series

What is a power series?

What is one power series that we are already familiar with?

Describe the different terms in the power series:

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

Describe some polynomial approximations of this power series, and then try to find some way to explain what these mean in the greater idea of a power series. You have some extra space here to draw our any graphs if you'd like.

Example For what values of x do the power series converge?

a. $\Sigma_{n=1}^{\infty}(-1)^{n-1}\left(\frac{x^n}{n}\right)$

b. $\Sigma_{n=0}^{\infty} n!\, x^{2n}$

c. $\Sigma_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)$

State the Convergence Theorem for Power Series

What is a corollary to this theorem?

Explain the steps for how to test a power series for convergence:

Example For each of the following, i) find the series interval of convergence, ii) determine for what values the series converges absolutely, and iii) determine for what values the series converges conditionally.

a. $\Sigma_{n=0}^{\infty}(x+8)^n$

b. $\Sigma_{n=1}^{\infty}\left(\frac{(x-3)^n}{n^4 4^n}\right)$

Lesson: Operations on Power Series

Explain series multiplication for power series

Give an example of how this works:

State the next theorem in this video

Explain term by term differentiation for power series:

Example Find $f'(x)$ and $f''(x)$ if $f(x) = \frac{1}{x-1} = \Sigma_{n=0}^{\infty} x^n, -1 < x < 1$

Explain term by term integration for power series:

Example Show how to differentiate and integrate term by term with the function: $\Sigma_{n=0}^{\infty}\left((-1)^n\left(\frac{x^{2n+1}}{3n+1}\right)\right)$

Examples: Intervals of Convergence

Find the series' radius and interval of convergence.

a. $\sum_{n=1}^{\infty} \frac{x^n}{n*5^n*\sqrt{n}}$

b. $\sum_{n=1}^{\infty} x^n (\ln n)$

c. $\sum_{n=2}^{\infty} \frac{x^n}{n\,(\ln n)^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{2*4*6*\ldots*2n}\ x^n$

Lesson: Convergence of Taylor Series

State **Taylor's Theorem**

State **Taylor's Formula**

What is another way to think of Taylor's formula?

What happens if $R_n(x) \to 0$ as $n \to \infty$?

Example Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for all values of x.

State **The Remainder Estimation Theorem**

Example Show that the Taylor Series for $\sin x$ at $x = 0$ converges for every value of x.

In the next examples, we will see how to use addition, subtraction, and multiplication with Taylor series. All of these operations are valid on the intersection of their intervals of convergence.

Example Use known series to find the first few terms of the Taylor series for the given function.

a. $\frac{1}{2}(3x + x\cos x)$

b. $x^2 \sin x$

Example For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ and obtain an error who magnitude is no greater than $3 - 10^{-4}$?

Lesson: Some Common Taylor Series

List the common Taylor series that you should have memorized:

Examples: Finding Taylor Series with Substitution

Use substitution to find the Taylor series at $x = 0$ of the functions below.

a. e^{-3x}

b. $\cos 7x^2$

Examples: Finding Taylor Series with Substitution

Use power series operations to find the Taylor series at $x = 0$ of the functions below.

a. $\sin x - x + \frac{x^3}{3!}$

b. $\sin x * \cos x$

c. $\sin^2 x$

Lesson: Introduction to Taylor and Maclaurin Series

Recall our power series:

$\Sigma_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$

Consider term-by-term differentiation for this series:

It we plug in x=a, what is the result of our derivatives?

From here, what result do we get?

Define a Taylor series generated by f(x) at x=a:

Explain what a Maclaurin series is:

Example Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $x = 2$.

Define a Taylor polynomial of order n:

<u>Example</u> Find the Taylor polynomials of orders 0, 1, 2, and 3 by: $f(x) = \frac{1}{x+1}$ at $x = 0$

<u>Example</u> Find the Taylor series and Taylor polynomials generated by $f(x) = e^x$ at $x = 0$

Example Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$

Examples: Finding Taylor Polynomials

Find the Taylor polynomials of orders 0, 1, 2, and 3 for $f(x) = \ln(1 + x)$ at $a = 0$.

Examples: Finding Taylor Series at x=0 (Maclaurin Series)

Find the Maclaurin series for the following:

a. xe^x

b. $\sinh x$

c. $x^4 - x^3 - 2x + 3$

Examples: Finding Taylor Series

Find the Taylor series generated by each function at $x = a$

a. $f(x) = x^4 + 2x^2 + 1, a = -2$

b. $f(x) = \frac{1}{x^2}, a = 1$

Examples: Maclaurin Series and Absolute Convergence

Find the first three nonzero terms of the Maclaurin series for the function $f(x) = (\sin x)(\ln(1 + x))$ and the values of x for which the series converges absolutely.

Lesson: The Binomial Series for Powers and Roots

Determine the Taylor series generated by: $f(x) = (1 + x)^n$

What is this series called?

When does the binomial series converge?

What is another way we can define the binomial series?

Example Determine the first 4 terms of the binomial series for the function $f(x) = \left(1 + \frac{x}{2}\right)^{-2}$

Lesson: Evaluating Nonelementary Integrals

We can use a familiar Taylor series to find the sum of given power series. For instance, how could we express:

$\int \cos x^2 \, dx$ as a power series?

How can we estimate $\int_0^1 \cos x^2 \, dx$ with an error less than 0.001.

Example Find a polynomial that will approximate $F(x)$ in the given interval with an error of magnitude less than 10^{-3}:

$F(x) = \int_0^x t^2 e^{-t^2} dt, [0, 1]$

Lesson: Evaluating Indeterminate Forms

We can evaluate indeterminate forms by expressing the functions involved as Taylor series.

Example Evaluate the limits:

a. $\lim\limits_{x \to 0} \frac{\sin x - x + (\frac{x^3}{6})}{x^5}$

b. $\lim_{x \to 2} \left(\frac{x^2 - 4}{\ln(x-1)} \right)$

Lesson: Euler's Identity

Review the powers of i:

Explain what $e^{i\theta}$ is expanded:

What result does this give us?

What is Euler's identity?

Example Write the power in the form $a + bi$: $e^{\frac{i\pi}{4}}$

Examples: Approximations and Nonelementary Integrals

a. Use series to estimate the integrals' value with an error of magnitude less than 10^{-5}:

$$\int_0^{0.4} \frac{e^{-x} - 1}{x} dx$$

b. Use series to approximate the values of the integral with an error of magnitude less than 10^{-8}:

$$\int_0^{0.1} e^{-x^2} dx$$

Examples: Indeterminate Forms

Use a series to evaluate the limits.

a. $\lim_{x\to 0} \frac{(e^x - e^{-x})}{x}$

b. $\lim_{x\to \infty} (x+1)\cos\frac{1}{x+1}$

Lesson: Parametrization of Plane Curves

Write out the vocabulary for parametrization:

Example Sketch the curve defined by the parametric equations.

a. $x = \cos\frac{\pi t}{4}, y = t, 0 \leq t \leq 8$

b. $x = t^2, y = t - 2, -\infty < t < \infty$

<u>Example</u> The position $P(x, y)$ of a particle moving in the xy-plane is given by $x = 2t - 5, y = 2t, 0 \leq t \leq 1$. Identify the Cartesian equation for its path and graph the equation. Indicate the portion of the graph traced by the particle and the direction of motion.

Example The position $P(x, y)$ of a particle moving in the xy-plane is given by $x = \frac{t}{t-1}, y = \frac{t-2}{t+1}, -1 < t < 1$. Identify the Cartesian equation for its path and graph the equation. Indicate the portion of the graph traced by the particle and the direction of motion.

How can any function be parametrized? Given an example.

<u>Example</u> Find a parametrization for the line through the point (a, b) with slope m.

Lesson: Tangents and Areas

When is a parametrized curve differentiable?

State the parametric formula for $\frac{dy}{dx}$:

State the parametric formula for $\frac{d^2y}{dx^2}$:

<u>Example</u> Find the tangent to the curve at the given value of t:

$x = 2\sin 2\pi t,\ y = \cos 2\pi t,\ t = -\frac{1}{6}$

Example Find the value of $\frac{d^2y}{dx^2}$ at the point t:

$x = 2\sin 2\pi t,\ y = \cos 2\pi t,\ t = -\frac{1}{6}$

Example Assuming that the equations define x and y implicitly as differentiable functions, find the slope of the curve at the given value of t:

$x = t^3 + t, y + 2t^3 = 2x + t^2, t = 1$

<u>Example</u> Find the area enclosed by the y-axis and the curve:

$x = t - t^2, y = 1 + e^{-t}$

Summary: Derivative Rules

For a derivatives exam, you will obviously need to know many formulas. To begin, we will summarize all of these formulas.

The formula for a derivative according to the definition:

The alternative formula for a derivative:

Compare using the formulas by taking the derivative of this function: $f(x) = x^2 - 3x$

Next we will review all of the derivative rules.

Basic Rules

Notes: c stands for a constant, n is any number, g(x) and h(x) are functions

For each function form, state what the derivative would be.

Derivative of a constant function: $f(x) = c; \frac{d}{dx}(c) =$

Power rule: $f(x) = x^n; \frac{d}{dx}(x^n) =$

Derivative constant multiple rule: $f(x) = cg(x); \frac{d}{dx}\left(cg(x)\right) =$

Derivative sum rule: $f(x) = g(x) + h(x); \frac{d}{dx}\left(g(x) + h(x)\right) =$

Derivative of the natural exponential function: $f(x) = e^x, \frac{d}{dx}(e^x) =$

Derivative product rule: $f(x) = g(x) * h(x), \frac{d}{dx}\left(g(x) * h(x)\right) =$

Derivative quotient rule $f(x) = \frac{g(x)}{h(x)}\ (h(x) \neq 0); \frac{d}{dx}\left(\frac{g(x)}{h(x)}\right) =$

Examples Find the derivatives of the functions.

a. $f(x) = \frac{e^x}{\sqrt{x} - 4x^2}$

b. $y = 5x^\pi e^x$

Trig Rules and the Chain Rule

Derivative of sine: $f(x) = \sin x\,; \frac{d}{dx}(\sin x) =$

Derivative of cosine: $f(x) = \cos x\,, \frac{d}{dx}(\cos x) =$

Derivative of tangent: $f(x) = \tan x\,, \frac{d}{dx}(\tan x) =$

Derivative of cotangent: $f(x) = \cot x\,, \frac{d}{dx}(\cot x) =$

Derivative of secant: $f(x) = \sec x\,, \frac{d}{dx}(\sec x) =$

Derivative of cosecant: $f(x) = \csc x\,, \frac{d}{dx}(\csc x) =$

Chain rule: $f(x) = g\big(h(x)\big); \frac{d}{dx}\Big(g\big(h(x)\big)\Big) =$

Examples Find the derivatives of the functions.

a. $f(x) = \sin x\, e^x$

b. $g(x) = \frac{e^x}{\tan x}$

c. $y = \sin(e^{3x+1})$

d. $h(x) = \sec(sin\, x + (3x^2 + 5)^{18})$

Derivatives of Logarithms and Exponentials

Derivative of natural logarithm: $f(x) = \ln x; \frac{d}{dx}(\ln x) =$

Derivative of general exponential function: $f(x) = a^x, a > 0, a \neq 1; \frac{d}{dx}(a^x) =$

Derivative of general logarithm: $f(x) = \log_a x,\ a > 0, a \neq 1; \frac{d}{dx}(\log_a x) =$

Derivatives of Inverse Trig Functions

Derivative of inverse sine: $f(x) = \sin^{-1} x\,; \frac{d}{dx}(\sin^{-1} x) =$

Derivative of inverse cosine: $f(x) = \cos^{-1} x; \frac{d}{dx}(\cos^{-1} x) =$

Derivative of inverse tangent: $f(x) = \tan^{-1} x; \frac{d}{dx}(\tan^{-1} x) =$

Derivative of inverse cotangent: $f(x) = \cot^{-1} x; \frac{d}{dx}(\cot^{-1} x) =$

Derivative of inverse secant: $f(x) = \sec^{-1} x; \frac{d}{dx}(\sec^{-1} x) =$

Derivative of inverse cosecant: $f(x) = \csc^{-1} x; \frac{d}{dx}(\csc^{-1} x) =$

Examples Find the derivatives of the functions.

a. $f(x) = \log_3 x^5$

b. $y = \ln(sin(4^x))$

c. $g(x) = \sin^{-1}(\sqrt{x})$

d. $y = \csc^{-1}(\sin x)$

Suggestion: for each rule, make 2 of your own exercises and take the derivatives. Use a derivative calculator on Google to check your work!

Implicit Differentiation

Examples Find the derivatives of the functions.

a. $x^2 - xy + y^2 = 4$

b. $x^2 + y\sin x = x\cos 4y$

Logarithmic Differentiation

Why would you want to use this?

Examples Let's review a few examples.

a. $y = \frac{x^2\sqrt{(4x^2-2)^5}}{\sqrt[3]{x^3-1}}$

b. $y^{lnx} = x^y$

Logarithms and Differentiation

Sometimes we will want to use properties of logarithms to help use simplify functions so that we can take the derivatives of them. This is not logarithmic differentiation, this is just leveraging properties of logarithms

Examples Take the derivatives of the following functions.

a. $f(x) = \ln\frac{x\sqrt{x+1}}{x-4}$

b. $y = \log_4 x - \log_4 x^3$

What are all of the different types of notation for the following?

- First derivatives
- Second derivatives
- Third derivatives
- Fourth derivatives and higher derivatives

Examples Find the fourth derivative of the following function:

$$f(x) = \sin 3x + x^3 - e^{-2x} + 7$$

Examples Find the second derivative of the following function:

$$x^3 - y^4 = 5$$

What are 4 different interpretations for the derivative at a point?

Example Find the equation of the line tangent to $f(x) = x^2 - 3x + 1$ at the point $(1, -1)$.

Example A function has the derivative $f'(x) = 3x^2 - 2x$.

a. Do you have enough information to determine the equation of the tangent line at the point $x = 2$? If so, find the equation of the line.

b. What is the slope of the tangent line at $x = -3$?

c. What is a potential form of the original function?

d. If you know that $(-1, -4)$ is a point on $f(x)$, can you determine the original function? If so, find it.

Example Find the equation of the line that is tangent to $f(x) = 3x^2 - 4x$ and parallel to $y = 14x - 2$

Example At time t, the position of a body moving along the s-axis is $s = t^3 + 6t^2 + 9t$ ft.

a. What is the velocity of the object at a $t = 3$ seconds?

b. When will the object reach a velocity of zero?

d. Find the body's acceleration at a time t.

e. What is the speed of the object at $t = 3$ seconds?

Lesson: Length of Parametrically Defined Curves

How can we find the length of a curve C with parametric equations?

Example Find the length of the curve defined by: $x = \frac{t^2}{2}, y = \frac{(2t+1)^{\frac{3}{2}}}{3}, 0 \leq t \leq 4$

How does the formula apply for a curve defined by $y = f(x)$? Give an example.

Explain the arc length differential:

Example Find the coordinates of the centroid of the curve:

$$x = \cos t + t\sin t\,, y = \sin t - t\cos t\,, 0 \leq t \leq \frac{\pi}{2}$$

Lesson: Areas of Surfaces of Revolution from Parametric Curves

State the area of surface of revolution for parametrized curves:

Example Find the areas of the surfaces generated by revolving the curves:

$$x = \left(\frac{2}{3}\right)t^{\frac{3}{2}}, y = 2\sqrt{t}, 0 \leq t \leq \sqrt{3}$$

Lesson: Polar Coordinates

How do you define polar coordinates?

<u>Example</u> Find all the polar coordinates of the point $\left(3, \frac{\pi}{4}\right)$

<u>Example</u> Graph the following:

a. $2 \leq r \leq 3$ and $\theta = \frac{\pi}{4}$

<u>Example</u> Graph the following:

b. $-1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{4}$

c. $\frac{\pi}{4} \leq \theta \leq \frac{5\pi}{6}$

d. $r \geq 1$

How can you relate polar and Cartesian coordinates?

<u>Example</u> Rewrite as an equivalent Cartesian equation:

a. $r\cos\theta + r\sin\theta = 1$

b. $\cos^2\theta = \sin^2\theta$

Example Rewrite as an equivalent polar equation:

a. $x - y = 5$

b. $xy = 2$

Lesson: Graphing Polar Coordinate Equations

Tests for Symmetry for Polar Graphs in the Cartesian xy-Plane

<u>Example</u> Identify any symmetries of the curves.

a. $r = 2 + \sin\theta$

b. $r^2 = \sin\theta$

How to find the slope of the curve $r = f(\theta)$

<u>Example</u> Find the slopes of the curve at the given points.

a. $r = -1 + \sin\theta$ at $\theta = 0, \pi$

b. $r = \cos 2\theta$, $\theta = 0, \pm\frac{\pi}{2}$

How can you graph a polar equation in the Cartesian xy-plane?

Example Graph the following in the Cartesian xy-plane

a. $r = 2 + \sin\theta$

b. $r^2 = \sin\theta$

Lesson: Areas in Polar Coordinates

Explain how to find areas with polar equations:

Example Find the areas of the regions:

a. Bounded by the circle $r = 3\sin\theta$ for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

d. Inside the circle $r = 4$ and above the line $r = 2\csc\theta$

Lesson: Lengths of a Polar Curve

How do you find the length of a polar curve?

<u>Example</u> Find the length of the curves:

a. The spiral $r = \frac{e^{\theta}}{\sqrt{2}}, 0 \leq \theta \leq \pi$

b. The parabolic segment $r = \frac{4}{1+\cos\theta}$, $0 \leq \theta \leq \frac{\pi}{2}$